Extra Exercises for 1/26

1. Show that if p is an odd prime and $a \in \mathbb{Z}$ is such that $p \nmid a$ and $k \geq 1$ then

$$a^{p^{k-1}(p-1)} \equiv 1 \bmod p^k.$$

2. Prove under the hyptheses in 1. that

$$\frac{a^{p^{k-1}(p-1)} - 1}{p^k} \equiv \frac{a^{p-1} - 1}{p} \bmod p.$$

3. Let $\zeta = e^{2\pi i/3}$. Find $f(t) = a_0 + a_1 t + a_2 t^2 + t^3$ with $a_i \in \mathbb{Z}$ for $i = 0, 1, 2, 3$ such that $f(1 + \zeta) = 0$. Is there another monic (coefficient of the coefficient of the hishest power of t) polynomial of degree at most 3 with integer coefficients, $g(t)$, such that $g(1 + \zeta) = 0$?