We note that with our definition of Vinberg pair the group G^{θ} is connected. This follows from

Theorem. If K is a connected, simply connected compact Lie group and if θ is an automorphism of K then K^{θ} is connected.

Corollary. If (H,V) is a Vinberg pair then G^{θ} is connected.

Proof. We have seen that we may assume that G is a symmetric subgroup of $GL(\mathfrak{g})$ with an appropriate inner product $\langle,\rangle$. We also showed that we may assume that

$$K = U(\mathfrak{g},\langle,\rangle) \cap G$$

is invariant under θ. Weyl's theorem implies that the connecte, simply connected covering group, $\tilde{K}$, of K is compact. Thus $\tilde{K}^{\theta}$ is connected. Hence

$$K^{\theta} = Ad_{\mathfrak{g}}(\tilde{K}^{\theta})$$

is connected. Now

$$G^{\theta} = K^{\theta}\,\mathsf{exp}(iLie(K^{\theta})).$$

Hence G^{θ} is connected. ■

If L is an m–dimensional algebraic torus then set $\hat{L}$ equal to the set of regular characters. $\hat{L}$ is a group under point-wise multiplication.

Lemma. Let L be an algebraic torus of dimension m. Then there exist elements $\eta_1, ..., \eta_m \in \hat{L}$ such that the map

$$\mathbb{Z}^n \to \hat{L}$$

$$J \longmapsto \eta_1^{j_1} \cdots \eta_m^{j_m}$$

is a group isomorphism.

We use the notation $T_{\mathfrak{a}}$ for the intersection of all Zariski closed subgroups, U, of G such that Lie$(U) \supset \mathfrak{a}$. That is $T_{\mathfrak{a}}$ is the Z–closure of exp$(\mathfrak{a})$.

Lemma. $T_{\mathfrak{a}}$ is an algebraic torus that is contained in the identity component of the center of the centralizer of $\mathfrak{a}$ in G.

Proof. Consider the Z–closure, T_1, of $\exp(\mathfrak{a})$. Then T_1 is abelian and it is contained in every Z–closed subgroup that whose Lie algebra contains $\mathfrak{a}$. Thus since the Lie algebra of a Lie group is the same as the Lie algebra of its identity component we see that $T_1 = T$ and T is connected and abelian. Let G_1 be the centralizer of $\mathfrak{a}$ in G then G_1 is reductive furthermore the center C of G_1 has the property that $\mathrm{Lie}(C) \supset \mathfrak{a}$. Hence T is contained in the identity component of the center of G_1. Thus, T consists of semi-simple elements and is connected so it is an algebraic torus. ∎

Proposition. $\dim T_{\mathfrak{a}} = \varphi(m) \dim \mathfrak{a}$ with $\varphi(m)$ the Euler Totient function (the number of $0 < j < m$ with $\gcd(j, m) = 1$).

Proof. Set $T = T_{\mathfrak{a}}$. Then $\theta : T \to T$ is an algebraic automorphism. The character group, $\hat{T}$, of T is a free abelian group of rank equal to $\mathbf{dim}\, T$. Thus if we identify a character with its differential we find that $\mathrm{Lie}(T)^*$ has a basis $\lambda_1, ..., \lambda_d$ such that θ^* has an integral matrix invertible. Let

$$e_1, ..., e_d$$

be the dual basis to

$$\lambda_1, ..., \lambda_d$$

then

$$\theta e_i = \sum a_{ji} e_j$$

with

$$a_{ij} \in \mathbb{Z}.$$

The characteristic polynomial, $f(x)$, of $\theta_{|\mathrm{Lie}(T)}$ is

$$f(x) = \sum_{k=0}^{d} c_k x^k$$

with

$$c_j \in \mathbb{Z}.$$

But since $\theta^m = 1$ we have a factorization over $\mathbb{Q}(\xi)$

$$f(x) = \prod_{k=0}^{m-1} (x - \xi^k)^{d_k}.$$

By definition of $T_{\mathfrak{a}}$ and the maximality of $\mathfrak{a}$, $d_1 = \dim \mathfrak{a}$. If

$$0 < j < m \text{ and } \gcd(j, m) = 1,$$

let

$$\sigma : \mathbb{Q}(\zeta) \to \mathbb{Q}(\zeta)$$

be the element of the Galois group $\mathbb{Q}(\xi)$ over $\mathbb{Q}$ defined by

$$\sigma(\xi) = \xi^j$$

We have if $x \in \mathbb{Q}(\xi)$

$$f(\sigma(x)) = \sum_{k=0}^{d} c_k \sigma(x)^k = \sigma(\sum_{k=0}^{d} c_k x^k)$$

also

$$\sigma(f(x)) = \prod_{k=0}^{m-1} (\sigma(x) - \xi^{kj})^{d_k}$$

and

$$f(\sigma(x)) = \prod_{k=0}^{m-1} (\sigma(x) - \xi^k)^{d_k}.$$

This implies that

$$d_j = d_1 \text{ if } 0 < j < m \text{ and } \gcd(j, m) = 1.$$

Hence,

$$\dim \mathrm{Lie}(T) \cap \mathfrak{g}_{\zeta^j} = \dim \mathfrak{a}.$$

This yields the lower bound

$$\dim T \geq \varphi(m)\mathfrak{a}.$$

To prove the upper bound we show that if

$$\mathfrak{b} = \sum_{\gcd(m,j)=1} \mathrm{Lie}(T) \cap \mathfrak{g}_{\zeta^j}$$

then

$$\exp(\mathfrak{b}) = T.$$

We note that if h is the m–th cyclotomic polynomial then

$$\mathfrak{b} = \ker h(\theta_{|\mathrm{Lie}(T)}).$$

Let x be an indeterminate and

$$h(x) = \sum_{j=0}^{\varphi(m)-1} r_j x^j.$$

As is standard, $r_j \in \mathbb{Z}$. We consider the regular homomorphism of T to T given by

$$\beta(z) = \prod_{j=0}^{\phi(m)-1} \theta(z)^{r_j}.$$

Then

$$d\beta = h(\theta_{|\mathsf{Lie}(T)})$$

and

$$Lie\,\mathsf{ker}\,\beta = \mathsf{ker}\,d\beta$$

Hence the identity component of $\mathsf{ker}\,\beta$ is an algebraic subtorus of T whose Lie algebra contains $\mathfrak{a}$. This implies that $T = \mathsf{exp}\,\mathfrak{b}$ and so $\mathsf{dim}\,T \leq \varphi(m)\,\mathsf{dim}\,\mathfrak{a}$. ■

Exercise. Show that if $m = 2$ then $T_{\mathfrak{a}} = \mathsf{exp}(\mathfrak{a})$.

If G is a Lie group, H is a closed subgroup and $\mathfrak{a}$ is a subspace of Lie(G) then the normalizer of $\mathfrak{a}$ in H is the subgroup

$$N_H(\mathfrak{a}) = \{g \in H | Ad(g)\mathfrak{a} \subset \mathfrak{a}\}$$

and the centralizer of $\mathfrak{a}$ is the subgroup

$$C_H(\mathfrak{a}) = \{g \in H | Ad(g)x = x, x \in \mathfrak{a}\}.$$

The Weyl group of $\mathfrak{a}$ is the group

$$W_H(\mathfrak{a}) = N_H(\mathfrak{a})/C_H(\mathfrak{a})$$

which we will think of as a group of linear maps of $\mathfrak{a}$ to $\mathfrak{a}$.

Lemma. If ad$x$ is diagonalizable for every element in $\mathfrak{a}$ and $[x, y] = 0, x, y \in \mathfrak{a}$ then Lie$(N_H(\mathfrak{a}))$ = Lie$(C_H(\mathfrak{a}))$.

Proof. We note that

$$\text{Lie}(N_H(\mathfrak{a})) = \{x \in \text{Lie}(H) | [x, \mathfrak{a}] \subset \mathfrak{a}\}$$

and

$$\text{Lie}(C_H(\mathfrak{a})) = \{x \in \text{Lie}(H) | [x, \mathfrak{a}] = 0\}.$$

If $x \in \mathsf{Lie}(N_H(\mathfrak{a}))$ and $y \in \mathfrak{a}$ then $[x,y] \in \mathfrak{a}$ so $(\mathsf{ad}y)^2 x = 0$. This implies that $[y,x] = 0$ since $\mathsf{ad}y$ is diagonalizable. Hence $y \in \mathsf{Lie}(C_H(\mathfrak{a}))$. ∎

Corollary. If $\mathfrak{a}$ satisfies the conditions of the previous lemma and if G is a linear algebraic group and H is a Z–closed subgroup then $W_H(\mathfrak{a})$ is finite.

Proof. The previous lemma implies that $C_H(\mathfrak{a})$ contains the identity component of $N_H(\mathfrak{a})$ since an algebraic group has a finite number of connected components the result follows. ∎

Corollary. If (H,V) is a Vinberg pair and if $\mathfrak{a}$ is a Cartan subspace of V then $W_H(\mathfrak{a})$ is finite.

We note that since any two Cartan spaces of V are conjugate by H the Weyl group is, up to isomorphism, independent of the choice of $\mathfrak{a}$. So we call it the Weyl group of the Vinberg pair.

Proposition. If $x, y \in \mathfrak{a}$, a Cartan subspace of V then $Hx = Hy$ if and only if $W_H(\mathfrak{a})x = W_H(\mathfrak{a})y$.

Proof. We assume $x, y \in \mathfrak{a}$ and that there exists $h \in H$ with $hx = y$. Let $g \in \left(G^\theta\right)^o$ be such that $g_{|V} = h$. We consider the Vinberg pair

$$\left(\left([C_G(y), C_G(y)]^\theta\right)^o_{|\mathsf{Lie}([C_G(y),C_G(y)])\cap V}, \mathsf{Lie}([C_G(y), C_G(y)])\right)$$

Here if $\mathfrak{z}_1$ is the center of $C_G(y)$ then $\mathfrak{a} = \mathfrak{z}_1 \oplus \mathfrak{a}_1$ and $\mathfrak{a}_1$ and $g\mathfrak{a}_1$ are Cartan subspaces of $\mathsf{Lie}([C_G(y), C_G(y)]) \cap V$. Hence there exists u in the identity component of $[C_G(y), C_G(y)]^\theta$ such that

$$u\,(g\mathfrak{a}_1) = \mathfrak{a}_1.$$

Since $u\mathfrak{z} = \mathfrak{z}$ we see that

$$ug_{|V} \in N_H(\mathfrak{a})$$

and

$$ugx = y.$$

The converse is obvious. ■

We now come to Vinberg's generalization of the Chevalley restriction theorem.

Theorem 1 *The restriction map* $\mathrm{res}_{V/\mathfrak{a}} : \mathcal{O}(V)^H \to \mathcal{O}(\mathfrak{a})^{W_H}$ *is an isomorphism of algebras.*

Proof. Set $W = W_H(\mathfrak{a})$. We first observe

$$B = \mathrm{res}_{V/\mathfrak{a}}(\mathcal{O}(V)^H)$$

and let $q(B)$ be the quotient field of B. Let F be the field of rational functions on $\mathfrak{a}$. Then F is the quotient field of $\mathcal{O}(\mathfrak{a})$. We assert that

F is a normal extension of $q(B)$.

Indeed, set

$$\det(tI - \mathrm{ad}X) = \sum_{j=0}^{p} t^j D_{p-j}(X)$$

for $X \in V$ (the $\mathrm{ad}X$ is the adjoint action comming from $\mathfrak{g}$) then $D_{p-j} \in \mathcal{O}(V)^H$ so $h(t) = \sum_{j=0}^{p} t^j \mathrm{res}_{V/\mathfrak{a}} D_{p-j}(X)$ is in $q(B)[t]$. The roots of this polynomial consist of 0 and the elements of $\Sigma(\mathfrak{a})$. Since the span of $\Sigma(\mathfrak{a})$ is $\mathfrak{a}^*$ we see that F is the splitting field of $h(t)$ and thus a normal extension of $q(B)$. This proves 2.We note that it also proves that the elements of the Galois group $Gal(F/q(B))$ map $\mathfrak{a}^*$ to $\mathfrak{a}^*$ and thus preserve $\mathcal{O}(\mathfrak{a})$.

This implies that

$$q(B) = \{f \in F | \sigma f = f, \sigma \in Gal(F/q(B))\}.$$

Let for $\sigma \in Gal(F/q(B)), x \in \mathfrak{a}$

$$\delta_{\sigma,x}(f) = \sigma f(x).$$

This defines a homomorphism of $\mathcal{O}(\mathfrak{a})$ to $\mathbb{C}$. The Nullstellensatz implies that there exists $x_1 \in \mathfrak{a}$ such that $\delta_{\sigma,x}(f) = f(x_1)$ for all $f \in \mathcal{O}(\mathfrak{a})$. We note that

$$f(x_1) = f(x)$$

for all $f \in B$. So there exists $h \in H$ such that $x_1 = hx$. The previous Lemma implies that there exists $s \in W$ so that $x_1 = sx$. Hence if $f \in \mathcal{O}(\mathfrak{a})^W$ then $\sigma f = f$. This implies that $\mathcal{O}(\mathfrak{a})^W \subset B$. Since the converse is obvious the result follows. ∎

This result applies to $Ad(G)$ acting on $\mathfrak{g}$. We have seen that

$$W = W_G(\mathfrak{h})$$

is generated by reflections.

Theorem. Let

$$f_1, ..., f_l$$

be a minimal set of homogeneous generators for

$$\mathcal{O}(\mathfrak{h})^W.$$

There exist

$$u_1, ..., u_l$$

such that

$$u_{i|\mathfrak{h}} = f_i$$

and

$$\mathcal{O}(\mathfrak{g})^G = \mathbb{C}[u_1, ..., u_l].$$