At this point we have proved ($W \subset GL(n, \mathbb{C})$ then $R(W)$ is the set of complex reflections in W).

Lemma If W is a finite subgroup $GL(n, \mathbb{C})$ such that $\mathcal{O}(\mathbb{C}^n)^W$ has n, algebraically independent, homogeneous, generators of degrees $d_1, ..., d_n$ then

1. $d_1 \cdots d_n = |W|$

2. $|R(W)| = \sum_{j=1}^{n} \left(d_j - 1\right)$.

We are now ready to prove the enhancement of 1.$\Longrightarrow$ 2.

Assume that we have $g_1, ..., g_n$ algebraically independent homogeneous elements of $\mathcal{O}(V)^W$ with $\deg g_i = e_i$.

Molien's formula implies that

$$\frac{1}{|W|} \sum_{s \in W} \frac{1}{\det(1 - qs)} = \sum_{j \geq 0} q^j \dim \mathcal{O}^j(\mathbb{C}^n)^W.$$

Also

$$\dim \mathbb{C}^d[g_1, ..., g_n] = \left|\{I | \sum i_j e_j = d\}\right| = a_j.$$

Clearly $a_j \leq \dim \mathcal{O}^j(\mathbb{C}^n)^W$. Thus for $0 < q < 1$ we have

$$\frac{1}{|W|} \sum_{s \in W} \frac{(1-q)^n}{\det(1-qs)} \geq \prod_{j=1}^n \frac{1-q}{1-q^{e_j}}$$

taking the limit as $q \to 1$ we have

$$\frac{1}{|W|} \geq \frac{1}{e_1 \cdots e_n}.$$

Now suppose that $|W| = e_1 \cdots e_n$ and $|q| < 1$ then

$$\frac{1}{|W|} \sum_{s \in W - \{I\}} \frac{(1-q)^n}{\det(1-qs)} =$$

$$\frac{1}{|W|} \sum_{s \in W} \frac{(1-q)^n}{\det(1-qs)} - \frac{1}{|W|} \geq \prod_{j=1}^{n} \frac{1-q}{1-q^{e_j}} - \frac{1}{|W|}.$$

Hence

$$\frac{|R(W)|}{2} = \lim_{\substack{q \to 1 \\ q < 1}} \frac{\left(\sum_{s \in W} \frac{(1-q)^n}{\det(1-qs)} - 1 \right)}{1-q} \geq$$

$$\lim_{\substack{q \to 1 \\ q < 1}} \frac{\left(|W| \prod_{j=1}^{n} \frac{1-q}{1-q^{e_j}} - 1 \right)}{1-q} = \frac{1}{2} \sum_{i=1}^{n} (e_i - 1).$$

If W' is the subgroup generated by $R(W)$ then $\mathcal{O}(\mathbb{C}^n)^{W'}$ is generated by $f_1, ..., f_n$, homogeneous, algebraically independent elements of $\mathcal{O}(\mathbb{C}^n)^{W'}$. Since $\mathcal{O}(\mathbb{C}^n)^W \subset \mathcal{O}(\mathbb{C}^n)^{W'}$ if $d_i = \deg f_i$ and $d_1 \leq ... \leq d_n$ then

$$d_i \leq e_i, i = 1, ..., n.$$

However,

$$\frac{|R(W)|}{2} = \frac{|R(W')|}{2} = \frac{1}{2}\sum(d_i - 1).$$

This implies that

$$\frac{1}{2}\sum(d_i - 1) \geq \frac{1}{2}\sum_{i=1}^{n}(e_i - 1).$$

Hence $e_i = d_i$. Hence

$$\left|W'\right| = d_1 \cdots d_n = e_1 \cdots e_n = |W|.$$

So $W = W'$.Also since $e_i = d_i$ we see that $\mathbb{C}[g_1, ..., g_n] = \mathbb{C}[f_1, ..., f_n] = \mathcal{O}(\mathbb{C}^n)^W$.

Examples. 1. Taking $g_j = \sum_{i=1}^n x_i^j$. Then $g_1, ..., g_n$ are algebraically independent and $\deg g_j = j$. So

$$\mathcal{O}(\mathbb{C}^n)^{S_n} = \mathbb{C}[g_1, ..., g_n].$$

2.W the group of signed permutations acting on $\mathbb{C}^n$. Then $|W| = 2^n n!$. W is generated by the transpositions and the reflections, ε_i, about the hyperplanes $x_i = 0$. That is , ε_j is diagonal with diagonal entries 1 except in the i–th position and a -1 in the i–position.

Set $g_j = \sum_{i=1}^n x_i^{2j}$.Then $g_1, ..., g_n$ are algebraically independent and $\deg g_j = 2j$.Thus $\mathbb{C}[g_1, ..., g_n] = \mathcal{O}(\mathbb{C}^n)^W$.

3. W the group of signed permutations acting on $\mathbb{C}^n$ with an even number of sign changes. Let ε_i be the Then W is generated by the transpositions and the reflections $\varepsilon_i\varepsilon_j(ij)$ for $i \neq j$. Let $g_j = \sum_{i=1}^n x_i^{2j}$, $i = 1, ..., n-1$ and $g_n = x_1 \cdots x_n$. $e_1 \cdots e_n = 2^{n-1} n!$ hence $\mathbb{C}[g_1, ..., g_n] = \mathcal{O}(\mathbb{C}^n)^W$.

This describes generating sets of invariants for the classical Weyl groups. $W_{A_n}, W_{B_n}(= W_{C_n}), W_{D_n}$.

Chevalley's argument to prove 1. in the theorem.

W a finite subgroup of $GL(n, \mathbb{C})$

$u_1, ..., u_r$ be a minimal set of generators for the ideal $\mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)^W_+$.

$d_i = \deg u_i$ and $t_1, ..., t_r$ be indeterminates with assigned degrees $d_1, ..., d_r$.

φ a homogeneous polynomial of minimal positive weighted degree such that $\varphi(u_1, ..., u_r) = 0$.

Let $\varphi_i = \frac{\partial \varphi}{\partial t_i}$. If $\varphi_i(u_1, ..., u_r) = 0$ then $\varphi_i = 0$.

Let $\varphi_i \neq 0$ for $i \leq m$ and $\varphi_i = 0$ for $i > m$.

Set $q_i = \varphi_i(u_1, ..., u_r)$ for $i \leq m$. Then

$$0 = \frac{\partial \varphi(u_1, ..., u_r)}{\partial x_j} = \sum_{i=1}^{m} q_i \frac{\partial u_i}{\partial x_j}.$$

After reordering we assume that $q_1, ..., q_l$ form a minimal subset of $\{q_1, ..., q_m\}$ generating $\sum_{i=1}^{m} \mathcal{O}(\mathbb{C}^n) q_i$.

That us if $l < i \leq m$ then we have

$$q_i = \sum_{k=1}^{l} h_{ik} q_k$$

with $h_{ij} \in \mathcal{O}(\mathbb{C}^n)$ and homogenous.

Substituting

$$\sum_{i=1}^{l} q_i \frac{\partial u_i}{\partial x_j} + \sum_{i=l+1}^{m} \sum_{k=1}^{l} h_{ik} q_k \frac{\partial u_i}{\partial x_j} = 0.$$

This yields

$$\sum_{i=1}^{l} q_i \left(\frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \frac{\partial u_p}{\partial x_j} \right) = 0$$

If we set

$$v_{ij} = \frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \frac{\partial u_p}{\partial x_j}$$

then v_{ij} is homogeneous of degree $d_i - 1$ and so

$$q_i v_{ij} \in \sum_{k \neq i, k \leq l} \mathcal{O}(\mathbb{C}^n) q_k.$$

Suppose we could prove that this implies that $v_{ij} \in \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)_+^W$. Then

$$\frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \frac{\partial u_p}{\partial x_j} = \sum w_{ijk} u_k.$$

Noting that $i \leq l$ and we can discard all homogeneous terms degree $\neq d_i - 1$ we see that after throwing away those terms the right hand side is

$$\sum_{k \neq i} w_{ijk} u_k.$$

Multiply both sides of this equation by x_j and sum in j we have

$$\sum_j x_j \frac{\partial u_i}{\partial x_j} + \sum_{p=l+1}^{m} h_{pi} \sum_j x_j \frac{\partial u_p}{\partial x_j} = \sum_{k \neq i} \left(\sum_j x_j w_{ijk} \right) u_k.$$

If φ is homogeneous of degre k then

$$\sum_j x_j \frac{\partial f}{\partial x_j} = kf.$$

We therefore have

$$d_i u_i = - \sum_{p=l+1}^{m} h_{pi} d_p u_p + \sum_{k \neq i} \left(\sum_j x_j w_{ijk} \right) u_k.$$

This contradicts the minimality of $u_1, ..., u_r$.

Here is Chevalley's lemma which completes the proof.

Lemma. Let $W \subset GL(n, \mathbb{C})$ be a finite complex reflection group and let $u_1, ..., u_r$ be homogenous elements of $\mathcal{O}(\mathbb{C}^n)^W_+$. Suppose that $u \in \mathcal{O}(\mathbb{C}^n)^W_+$ and $v \in \mathcal{O}(\mathbb{C}^n)$ are homogenous and that $u \notin \sum \mathcal{O}(\mathbb{C}^n)u_i$ and that

$$uv \in \sum \mathcal{O}(\mathbb{C}^n)u_i$$

then $v \in \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)^W_+$.

s a complex reflection then $\dim \ker(I - s) = n - 1$ so $\ker(I - s) = \ker \lambda$ with $\lambda \in (\mathbb{C}^n)^*$.

Let $u \in \mathbb{C}^n$ satisfy $\lambda(u) = 1$ and $su = \zeta u$. If $x \in \mathbb{C}^n$ then $x - \lambda(x)u \in \ker \lambda$ so

$$sx = s(x - \lambda(x)u) + \lambda(x)su = x - \lambda(x)u + \zeta\lambda(x)u$$

That is

$$sx = x + (\zeta - 1)\lambda(x)u.$$

Exercise. If λ is in $(\mathbb{C}^n)^*$,if $u \in \mathbb{C}^n$ is such that $\lambda(u) = 1$ and ζ is a primitive m–th, root of 1 with $m \neq 1$. Defining

$$s_{\lambda,\zeta,u}(x) = x + (\zeta - 1)\lambda(x)u$$

then $s_{\lambda,\zeta}$ is a complex reflection.

Proof of Lemma by induction on degree v. If $\deg v = 0$ then v must be zero by the hypothesis.

If s is a complex reflection, such that $s = s_{\lambda,\zeta,u}$ then if $w \in \mathcal{O}(\mathbb{C}^n)$ then $(I - s^*)w = \lambda\tilde{w}$. If $h \in \mathcal{O}(\mathbb{C}^n)^W_+$ then $(1 - s^*)hw = hw - s^*hs^*w = h(I - s^*)w)$.So

$$u(I - s^*)v \in \sum ((1 - s^*)\mathcal{O}(\mathbb{C}^n))\, u_i$$

So both sides of the equation are divisble by λ thus

$$u\tilde{v} \in \sum \mathcal{O}(\mathbb{C}^n)u_i.$$

So the inductive hypothesis implies

$$\tilde{v} \in \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)^W_+.$$

Hence

$$(I - s^*)v \in \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)^W_+$$

for all complex reflections in W. This implies the same holds true for all $s \in W$. We therefore have after summing both sides over $s \in W$

$$|W|\, v \in \sum_{s \in W} s^*v + \mathcal{O}(\mathbb{C}^n)\mathcal{O}(\mathbb{C}^n)^W_+.$$

Note This gives another necessary and sufficient condition that for a finite subgroup of $GL(n, \mathbb{C})$ to be a finite reflection group.