13 A set of axioms for a quantum system

As mathematicians we want a well defined set of rules that will allow us to derive theorems about quantum systems subject to additional properties. However, we have seen in the previous sections that quantum systems do not exist in isolation and there are environments that cause decoherence and the need to study mixed states rather than pure states. We will thus give a set of axioms for an ideal system that is isolated from the environment. We should keep in mind that this apparent isolation will only persist for a short time interval. We will now begin our task.

A quantum system is a Hilbert space, $\mathcal{H}$, the quantum states of $\mathcal{H}$ are the elements of the projective space on $\mathcal{H}$, $\mathbb{P}(\mathcal{H})$. The states will be denoted $[v]$ with v a basis and usually we will choose v of norm 1. A measurement is a self adjoint operator on $\mathcal{H}$ as is a Hamiltonian.

1. If A is a measurement on $\mathcal{H}$ and $[v]$ is a state then the expected value of the measurement is

$$\frac{\langle Av|v\rangle}{\|v\|^2} = \frac{\langle v|A|v\rangle}{\|v\|^2}.$$

2. Let P_A be the projection valued measure associated with A. Then after the measurement of $[v]$ the outcome will be a real number that is in the set S with probability

$$\frac{\|P_A(S)v\|}{\|v\|^2}$$

and the state has collapsed to a state $[u]$ contained in $P_A(S)[v]$. If A diagonalizes on $\mathcal{H}$ then there is a counatble set $\{\lambda_j\}$ of eigenvalues and if P_j is the projection onto the λ_j eigenspace then we $P_A(S) = \sum_{\lambda_j \in S} P_j$, And this axiom can be simplified to a measurement yields an eigenvalue, λ_j, of A and the state collapses to $P_j[v]$.

3. The valid operations on states are unitary operators, measurements and after a measurement a classical operation.

4. The dynamics of the state, $[v]$, are given by a Hamiltonian, H, having the state in its domain and the state evolves in time according to the equation

$$i\hbar\frac{d}{dt}v = Hv.$$

As indicated above this is an idealized system and can only be taken to be accurate over short time intervals.we will now give an example of these axioms (all but the last) in a finite dimensional situation. Grover's algorithm gives a quantum method for unordered search that is the second most impressive algorithm in quantum computing.

Suppose we have a function from a set $\{0, 1, ..., N-1\}$ with N elements to $\{\pm 1\}$. This function takes the values

$$f(j) = (-1)^{\delta_{j,j_o}}$$

with j_o unknown. What we assume that we can ignore the time to calculate each value of the function. How many times must we read the function to know j_o. Obviously to know it prefectly we must make $N-1$ readings. To know it with probability p we must make pN readings. Grover cuts this down to a multiple of $\sqrt{N}$.

The algorithm makes use of two unitary operators on $\mathbb{C}^N$. Thought of as having basis $\{|j\rangle\}_{j=0}^{N-1}$. The first operator is defined by

$$T\,|j\rangle = f(j)\,|j\rangle$$

the second is a reflection about the uniform state

$$v_0 = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle\,.$$

Here if $\|v\| = 1$ then the reflection about v (really about the hyperplane orthogonal to v) is defined by

$$S_v w = w - 2\,\langle v|w\rangle\, v.$$

Note that $T = S_{|j_o\rangle}$. The second unitary operator is $S = S_{v_0}$. We set $U = ST$. The algorithm is

$$v_0 \to v_1\,(= Uv_0) \to v_2\,(= Uv_1) \to ... \to Uv_{k-1} \to ...$$

Using simple trigonometry we see that we should stop after

$$k = \left\lfloor \frac{\pi\sqrt{N}}{4} \right\rfloor$$

steps and measure relative to an operator with distinct eigenvalues with eigenvectors the given basis. If we do this measurement and if N is large then v_k will collapse to $[j_o]$ with probability 0.99...(some large number of nines).

To see this we note that if we set $e_1 = |j_o\rangle$ and

$$e_2 = \frac{v_0 - \langle v_o|e_1\rangle\, e_1}{\|v_0 - \langle v_o|e_1\rangle\, e_1\|}$$

then all of the action in the algorithm is in the vector space $\mathbb{R}e_1 + \mathbb{R}e_2$. If we write $v_0 = \cos(\theta)e_1 + \sin(\theta)e_2$ then if

$$R(\mu) = \begin{bmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{bmatrix}$$

then $v_0 = R(-\theta)e_1$ and $U^k = R(2k\theta)$. Thus

$$v_k = \cos((2k-1)\theta)e_1 - \sin((2k-1)\theta)e_2.$$

Exercise. Prove the above formulas. Also prove the probability assertion by observing that $\theta = \frac{\pi}{2} - \psi$ and $\cos\theta = \sin\psi$. Thus if N is large ψ is aproximately $\frac{1}{\sqrt{N}}$. Now do the trig.