Note Title 2/2/2011

Thm G locally compact topological group
let $\mathcal{B}$ be the σ-algebra generated by the open subsets
Then on $\mathcal{B}$ there is a unique countably additive left invariant, positive measure up to positive scalar.

If $g \in G$, μ is a Haar measure on G, locally compact group

$$R_g^* \mu = \mu \circ R_g^* \qquad R_g^* f = f \circ R_g$$

therefore implies $R_g^* \mu = c(g)\mu$.

$c: G \to \mathbb{R}_{>0}$ is constant.

$c(g) = \delta(g)^{-1}$, where δ is modular function of G

G Lie group, $X \in \mathrm{Lie}(G)$

$dR_g X_e = Y(g)_g \qquad Y(g) \in \mathrm{Lie}(G)$

$dL_{g^{-1}} dR_g X_e = Y(g)_e$

$\underbrace{(dL_{g^{-1}} \circ dR_g)}_{= \, d\mathrm{Int}(g^{-1})X} X = Y(g)$

$\mathrm{Int}(g)X = gXg^{-1}$

$d\mathrm{Int}(g) = \mathrm{Ad}(g)$

$\omega_g(X_{1g}, \ldots, X_{ng}) = \omega(X_1, \ldots, X_n)$, $\omega \in \Lambda^n \mathrm{Lie}(G)^* - \{0\}$

$R_g^* \omega = \det(\mathrm{Ad}(g)^{-1})\, \omega$

Corresponding measure μ is such that $R_g^* \mu = |\det(\mathrm{Ad}(g)^{-1})|\, \mu$

$\Rightarrow \delta(g) = |\det(\mathrm{Ad}(g))|$

G is said to be unimodular if $\delta(g) = 1$.

Examples: Observe $\delta: G \to \mathbb{R}_{>0}$ is a continuous gp. homomorphism

(1) If $[G, G] = G$, then $\delta = 1$

(2) If G is abelian, $\delta \equiv 1$ $(\because R_g = L_g)$

(3) If G is compact, $\delta \equiv 1$. $\delta: G \to \mathbb{R}_{>0}$ is cts gp. homo $(\because \delta(\mathbb{R}_{>0})$ is cpt$)$

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \;\middle|\; a>0,\ b\in\mathbb{R} \right\}$$

$L^2(G,\mu)$ $\quad \underline{\langle f_1, f_2\rangle} = \mu(\overline{f_2} f_1)$ $\qquad f_i \in C(G;\mathbb{C})$

A unitary representation of a locally compact topological group is a pair (π, H)
H a Hilbert space
and $\pi : G \to U(H) = \{ A : H \to H \mid A \text{ unitary operator} \}$
$\langle Av, Aw\rangle = \langle v, w\rangle \quad \forall v, w, \quad A$ bijective

$G \times H \to H$ given by
$(g, v) \mapsto \pi(g)v$ is continuous

(cts in strong top)
$\Leftarrow$ if $v \in H$, $\quad G \to H$ is continuous
$\qquad g \mapsto \pi(g)v$

E.g. If H is a finite dimensional Hilbert space and
$\pi : G \to U(H)$ gp homo
is cts in usual sense then it is a unitary rep and conversely.

Example $\quad H = L^2(G, \mu_L)$ fixed left inv. positive measure

$$\int_G f(g)\,dg = \mu_L(f)$$

$L^2(G)$, define $\pi(g) = L^*_{g^{-1}}$
$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$$
$$\langle \pi(g) f_1, \pi(g) f_2\rangle = \int_G f_1(g^{-1}x)\,\overline{f_2(g^{-1}x)}\,dx$$
$$\mu_L(\cancel{(g^{-1})^*} f_1 \overline{f_1})$$
$$= \langle f_1, f_2\rangle$$

$(\pi, L^2(G))$ is a unitary gp representation follows from the fact that a continuous functions on a compact set is uniformly continuous.

A unitary representation (π, H) of G is said to be irreducible if whenever $V \subseteq H$ is closed and $\pi(g)V \subseteq V$ $\forall g \in G$, then $V = \{0\}$ or $V = H$.

Thm If G is compact and $V \subseteq L^2(G)$ is closed and the action of G on V is irred. then $\dim V < \infty$.

Pf: let $f \in V$, f a unit vector

$$P_v = \langle v, f \rangle f$$

Define $\nu(v,w) = \int_G \langle \pi(g) P \pi(g)^{-1} v, w \rangle dg$

ν is sesquilinear on $V \times V$

i.e. linear in first variable, conjugate linear in second.

$$|\nu(v,w)| \leq C \|v\| \cdot \|w\|$$

$$\left(\because \quad |\langle \pi(g) P \pi(g)^{-1} v, w \rangle| \leq \|\cancel{\pi(g)} \cancel{P} \cancel{\pi(g)^{-1}} v\| \cdot \|w\| \right.$$

unitary projection

$$\left. \mu(|f|) \geq |\mu(f)| \right)$$

$\Rightarrow \exists Q : H \to H$ bounded s.t. $\langle Qv, w \rangle = \nu(v,w)$

$$Q\pi(g_1) = \pi(g_1) Q$$

$$\begin{aligned}
\langle \pi(g_1)^{-1} Q \pi(g_1) v, w \rangle &= \langle Q\pi(g_1) v, \pi(g_1) w \rangle \\
&= \int_G \langle \pi(g) P \pi(g)^{-1} \pi(g_1) v, \pi(g_1) w \rangle dg \\
&= \int_G \langle \pi(g_1^{-1} g) P \pi(g_1^{-1} g)^{-1} v, w \rangle dg \quad \text{left invariant} \\
&= \langle Qv, w \rangle
\end{aligned}$$

(1) $Q \neq 0$, $Q^* = Q$ self adjoint

(2) Q is a compact operator

- $Q: H \to H$ is compact means that if $B \subseteq H$ is bounded, then $\overline{QB}$ is compact.

- Spectral thm for compact self adjoint operator:

  H have countable orthogonal basis

  $T: H \to H$ self-adjont

  Then $H = H_0 \oplus \left(\overline{\bigoplus_{j=1}^{N} H_j} \right)$ $\quad 0 \le N < \infty$

  $\langle H_i, H_j \rangle = 0$ if $i \neq j$

  and $T|_{H_j} = \lambda_i I$ $\quad i \geq 0$ $\qquad \lambda_0 = 0$

  $\lambda_i \in \mathbb{R}$ and $\dim H_j < \infty$ for $j \geq 1$.

$\Rightarrow$ $\lambda_1 \neq 0$

$V_{\lambda_1} \neq 0$

$v = V_{\lambda_1}$, $Q\pi(g)v = \pi(g)Qv = \lambda_1 \pi(g) v$

$\Rightarrow \pi(g) V_{\lambda_1} \subseteq V_{\lambda_1}$ $\quad g \in G$

$\Rightarrow V_{\lambda_1} \neq 0$, $\quad V_{\lambda_1} = V_1$

$\Rightarrow \lambda_1 I$ is compact on V

$\Rightarrow$ V is finite dim.