

Homework Assigned 9/22

1. If the order of F is infinite then the assignment of $F[x_1, \dots, x_n]$ to functions on F^n by specialization is injective.

2. Show that an algebraically closed field is infinite.

Homework Assigned 9/25.

1. The Zariski topology on A^2 is not the product topology as $A^1 \times A^1$.

2. The map $f : A^1 \rightarrow X = \{(a, b) \in F^2 \mid b^2 - a^3 = 0\}$ given by $f(z) = (z^2, z^3)$ is a homeomorphism.

3. A closed subset of A^n is Hausdorff in the subspace topology if and only if it is finite. (This problem is harder than the others and involves material that will be developed in the next few classes.)

Homework Assigned 9/27.

1. There is no isomorphism between A^1 and the variety defined in problem 2. assigned on 9/25.

Homework Assigned 9/29.

p.32 1,6,7,8 Shafarevich.

Homework Assigned 10/2

1. Consider the variety $xy = 0$ go through the steps of the proof of the Noether normalization.

2. Let $a \in F$ find the irreducible components of $x^2 + y^2 + a = 0$.

*3. Find the irreducible components of the variety defined by $x^2 + y^2 + z^2 = 0$ and $x^2 - y^2 - z^2 + 1 = 0$.

Homework 10/4

1. Show that the map F to F given by $f(z) = z^2$ is a finite dominant morphism.

2. We look upon 4 dimensional affine space as the matrices

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

with entries $a_1, a_2, a_3, a_4 \in F$. We denote this affine variety by Y . If X is the matrix with indeterminate entries x_1, x_2, x_3, x_4 then we define $f(X) = X^2$ (the square of the matrix).

a) Show that f is dominant but not finite.

b) Let characteristic not 2. $u(X) = \det(X - (\text{tr} X)/2I)$ with I the identity matrix. Let $v(X) = u(X^2)$. Show that f is a finite mapping from $\mathbb{A}_{(v)}^4$ to $\mathbb{A}_{(u)}^4$ is dominant and finite. (This problem will be easier later in the quarter).

3. Let X be the variety defined by $y^2 - x^3 = 0$. Show that the map $f(z) = (z^2, z^3)$ is a finite morphism from \mathbb{A}^1 to X .

Homework 10/6 A, B are commutative algebras over F a field.

*1. The map $T : \text{Hom}_{F\text{-alg}}(A, F) \times \text{Hom}_{F\text{-alg}}(B, F) \rightarrow \text{Hom}_{F\text{-alg}}(A \otimes B, F)$ given by $T(f, g)(a \otimes b) = f(a)g(b)$ is bijective. Here \otimes is tensor product over F .

2. Show that if A and B are finitely generated algebras over an algebraically closed field without nonzero nilpotents then so is $A \otimes B$.

Homework 10/13 X an affine variety.

*1. Prove that X is irreducible if and only if any nonempty open subset of X is dense

2. Prove that X is irreducible if and only if any two non-empty open subsets of X have non-empty intersection.

Homework from 10/16 to 27

Read the first 19 pages in Kempf.

1. If X is an affine variety then if $f \in \mathcal{O}(X)$ then as a map from X to \mathbb{A}^1 , f is continuous.

2. If X is a topological space and $F = \mathbb{R}$ or \mathbb{C} with the usual metric topology and if we take for each open subset U the continuous functions from U to F . Then (X, \mathcal{O}_X) is a space with functions.

3. Let $F = \mathbb{R}$ and let $X = \mathbb{R}^n$ with the metric topology. If $U \subset X$ is open then define $\mathcal{O}_X(U)$ to be the space of all functions on U with continuous partial derivatives of all orders. Show that $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ is a space with functions. (Note one can define a smooth manifold of dimension n to be a space with functions that has an open covering by open subsets that are isomorphic with $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$ as a space of functions).

4. \mathbb{P}^1 is homeomorphic with \mathbb{A}^1 as a topological space.

*5. Prove that if $f = f(x_0, \dots, x_n)$ homogeneous of degree d then $\text{homog}_j(\text{dehomog}_j f) = x_j^r f$ for some r (depending on f and j). What is r ?

6. Write out the details in the proof that $K(\mathbb{P}^n)$ is isomorphic with $K(\mathbb{A}^n)$.

7. If X and Y are affine varieties then the notion of product is the same as the one we defined earlier.

8. If $F = \mathbb{R}$ or \mathbb{C} and if X and Y are topological spaces with structures of spaces with functions as in Exercise 2 let (Z, \mathcal{O}_Z) be a product of X and Y in the category of spaces with functions. Show that as a topological space Z is the topological product of X and Y . Show that $\mathcal{O}_Z(U)$ for U is not necessarily the algebra of continuous functions on U for U open in Z .

9. Exercise 1.6.2 in Kempf.

10. Exercise 1.6.4 in Kempf.

Homework 10/30

Read pp 41-57 in Shafarevich. The following three exercises are in pages 66 and 67 of Shafarevich:

1. Exercise 1 p.66 (here linear subspace means a subvariety defined by a set of linear=first degree equations).

2. Exercise 2.

3. Exercise 11.

4. If X and Y are projective varieties does a morphism pull back principal open subsets?

Homework November 1 to 17

*1. An open subgroup of an algebraic group is closed.

*2. If G is an algebraic group and H is a subgroup then the closure of H is a subgroup.

*3. Let G and H be algebraic groups show that if $f : G \rightarrow H$ is a morphism that is a group homomorphism then $f(H)$ is closed in H . (Hint: $f(H)$ has interior in its closure).

4. p.66 exercise 4 in Shafarevich.

5. If $X \subset \mathbb{A}^n$ is closed subset of dimension $n - k$ and X is the zero locus of k polynomials show that all irreducible components of X have dimension $n - k$.

Read Chapter 2 and 3 of Kempf and pp. 67 to 76 in Shafarevich do the following exercises on pp. 81-82:

1,4,11.

in Kempf do the following exercises on page 37.

(a),(b) (here a birational isomorphism is an isomorphism of an open dense subset onto an open dense subset) also a isomorphism over an open subset U is an isomorphism of U onto its image which is also open).

Homework November 27

*Let $f(x) \in F[x]$ be of degree $m > 0$ with m distinct roots and set $X = \mathbb{A}^2(y^2 - f(x))$ (i.e. the locus of zeros of the polynomial $y^2 - f(x)$).

1. Show that $y^2 - f(x)$ is irreducible in $F[x, y]$.

2. Show that X consists of nonsingular (smooth points).

3. Show that if $m = 1, 2$ then X is isomorphic with an open subset of \mathbb{A}^1 .

Identify \mathbb{A}^2 with $\mathbb{P}_0^2 = \{[1, a, b] | (a, b) \in \mathbb{A}^2\}$. Under this identification set Y equal to the closure of X in \mathbb{P}^2 .

4. Show that Y is the zero set of the homogenization of $y^2 - f(x)$.

Assume that $m > 2$.

5. Prove that $Y - X$ consists of one point and that if $m = 3$ then $Y \cap \mathbb{P}_j^2$ consists of nonsingular points for $j = 0, 1, 2$.