1. Consider $SL(3,\mathbb{C})$ acting on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ by the tensor product action ($g \longmapsto g \otimes g \otimes g$) show that any first degree, homogeneous polynomial invariant is the linear extension of

$$v_1 \otimes v_2 \otimes v_3 \longmapsto \det [v_1 v_2 v_3]$$

where v_i is considered a 3×1 column vector and $[v_1 v_2 v_3]$ is considered to be a 3×3 matrix.

2. Prove Lemma 12 on page 8 of the notes.

3. Let $\mathfrak{g}$ be a Lie subalgebra of $M_n(\mathbb{C})$ invariant under matrix adjoint and let $X \in \mathfrak{g}$ be diagonalizable. Prove that there is $P : M_n(\mathbb{C}) \to \mathfrak{g}_X = \{Y \in \mathfrak{g} | [Y, X] = 0\}$ a projection ($P^2 = P$) such that $ad(y)P = Pad(y)$ for all $y \in \mathfrak{g}_X$.

Hint: Let $(x, y) = tr(xy), x, y \in M_n(\mathbb{C})$ and let $Q : M_n(\mathbb{C}) \to \mathfrak{g}$ be the projection corresponding to

$$M_n(\mathbb{C}) = \mathfrak{g} \oplus \mathfrak{g}^{\perp}.$$

Show that adX is diagonalizable on $\mathfrak{g}$ and use this to see that there is a projection $P_1 : \mathfrak{g} \to \mathfrak{g}_X$ that satisfies $ad(y)P_1 = P_1 ad(y)$ for all $y \in \mathfrak{g}_X$.

4. Let $\mathfrak{g}$ be as in 3. Let $X \in \mathfrak{g}$ be diagonalizable and $Y \in \mathfrak{g}_X$ be nilpotent. Prove that there exists $H \in \mathfrak{g}_X$ such that $[H, Y] = Y$.

Hint: Using the Jordan form of Y in $M_n(\mathbb{C})$ we can conjugate Y to an element of the form

$$Z = gYg^{-1} = \begin{bmatrix} J_{n_1} & 0 & \cdots & 0 \\ 0 & J_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_k} \end{bmatrix}$$

with $n_1 \geq n_2 \geq .... \geq n_k > 0$ and $\sum n_j = n, g \in GL(n, \mathbb{C})$ and

$$J_r = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i.e. $r \times r$ with all entries 0 except for ones on the second diagonal). Let H_r be the diagonal matrix with diagonal entries $r, r-1, ..., 1$. Then if

$$U = \begin{bmatrix} H_{n_1} & 0 & \cdots & 0 \\ 0 & H_{n_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{n_k} \end{bmatrix}$$

then $[U, Z] = Z$. If $h = g^{-1}Ug$ then $[h, Y] = Y$. Now take $H = Ph$ with P as in the previous exercise.