Invariant theory and the measurement of quantum entanglement

Nolan R. Wallach

November 8, 2014

Quantum entanglement

- Two particles are entangled if a measurement of some property of one implies a well defined value of a corresponding property of the other.

Quantum entanglement

- Two particles are entangled if a measurement of some property of one implies a well defined value of a corresponding property of the other.
- Schrödinger's cat: A cat is in a box with an electron and a flask of poison gas that will be released if a device measures the spin of the electron to be down. Before the measurement the electron is spin up with a probability p and spin down with probability $1 - p$. Is the cat alive with probability p?

Quantum entanglement

- Two particles are entangled if a measurement of some property of one implies a well defined value of a corresponding property of the other.
- Schrödinger's cat: A cat is in a box with an electron and a flask of poison gas that will be released if a device measures the spin of the electron to be down. Before the measurement the electron is spin up with a probability p and spin down with probability $1 - p$. Is the cat alive with probability p?
- Einstein-Podolosky-Rosen: A pair of electrons are prepared with opposite spins. Basic quantum mechanics says that they will always have opposite spins. After some time one is measured and has spin up. This means that wherever the other is it will have spin down.

Quantum entanglement

- Two particles are entangled if a measurement of some property of one implies a well defined value of a corresponding property of the other.
- Schrödinger's cat: A cat is in a box with an electron and a flask of poison gas that will be released if a device measures the spin of the electron to be down. Before the measurement the electron is spin up with a probability p and spin down with probability $1 - p$. Is the cat alive with probability p?
- Einstein-Podolosky-Rosen: A pair of electrons are prepared with opposite spins. Basic quantum mechanics says that they will always have opposite spins. After some time one is measured and has spin up. This means that wherever the other is it will have spin down.
- Einstein: "Spooky action at a distance."

Pure State Entanglement

- We consider a Hilbert space $\mathcal{H}$(over $\mathbb{C}$) that is the tensor product of m Hilbert spaces
$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m.$$
The Hilbert spaces $\mathcal{H}_i$ will all be finite dimensional and usually 2 dimensional in this lecture.

Pure State Entanglement

- We consider a Hilbert space $\mathcal{H}$(over $\mathbb{C}$) that is the tensor product of m Hilbert spaces
$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m.$$
The Hilbert spaces $\mathcal{H}_i$ will all be finite dimensional and usually 2 dimensional in this lecture.
- A pure state in $\mathcal{H}$ is an element of the projective space on $\mathcal{H}$, $\mathbb{P}(\mathcal{H})$. This can be looked upon in two ways:

Pure State Entanglement

- We consider a Hilbert space $\mathcal{H}$(over $\mathbb{C}$) that is the tensor product of m Hilbert spaces
$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m.$$
The Hilbert spaces $\mathcal{H}_i$ will all be finite dimensional and usually 2 dimensional in this lecture.
- A pure state in $\mathcal{H}$ is an element of the projective space on $\mathcal{H}$, $\mathbb{P}(\mathcal{H})$. This can be looked upon in two ways:
- The set of unit vectors identified if they differ by a phase.

Pure State Entanglement

- We consider a Hilbert space $\mathcal{H}$(over $\mathbb{C}$) that is the tensor product of m Hilbert spaces
$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m.$$
The Hilbert spaces $\mathcal{H}_i$ will all be finite dimensional and usually 2 dimensional in this lecture.
- A pure state in $\mathcal{H}$ is an element of the projective space on $\mathcal{H}$, $\mathbb{P}(\mathcal{H})$. This can be looked upon in two ways:
- The set of unit vectors identified if they differ by a phase.
- The set of one dimensional subspaces.

Pure State Entanglement

- We consider a Hilbert space $\mathcal{H}$(over $\mathbb{C}$) that is the tensor product of m Hilbert spaces
$$\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m.$$
The Hilbert spaces $\mathcal{H}_i$ will all be finite dimensional and usually 2 dimensional in this lecture.
- A pure state in $\mathcal{H}$ is an element of the projective space on $\mathcal{H}$, $\mathbb{P}(\mathcal{H})$. This can be looked upon in two ways:
- The set of unit vectors identified if they differ by a phase.
- The set of one dimensional subspaces.
- Thus there is an action of $GL(\mathcal{H})$ on the states.

- A product state is a state that can be represented as a tensor product $\phi_1 \otimes \cdots \otimes \phi_m$ with ϕ_i a state in $\mathcal{H}_i$.

- A product state is a state that can be represented as a tensor product $\phi_1 \otimes \cdots \otimes \phi_m$ with ϕ_i a state in $\mathcal{H}_i$.
- The simplest algebraic definition of an entangled state is a state that is not a product state.

- A product state is a state that can be represented as a tensor product $\phi_1 \otimes \cdots \otimes \phi_m$ with ϕ_i a state in $\mathcal{H}_i$.
- The simplest algebraic definition of an entangled state is a state that is not a product state.
- If $m > 1$ then a randomly chosen state will be entangled Since the dimension of the set of states is $d_1 \cdots d_m - 1$ and the dimension of the set of product states is $d_1 + ... + d_m - m + 1$.

- A product state is a state that can be represented as a tensor product $\phi_1 \otimes \cdots \otimes \phi_m$ with ϕ_i a state in $\mathcal{H}_i$.
- The simplest algebraic definition of an entangled state is a state that is not a product state.
- If $m > 1$ then a randomly chosen state will be entangled Since the dimension of the set of states is $d_1 \cdots d_m - 1$ and the dimension of the set of product states is $d_1 + ... + d_m - m + 1$.
- Natural questions:

- A product state is a state that can be represented as a tensor product $\phi_1 \otimes \cdots \otimes \phi_m$ with ϕ_i a state in $\mathcal{H}_i$.
- The simplest algebraic definition of an entangled state is a state that is not a product state.
- If $m > 1$ then a randomly chosen state will be entangled Since the dimension of the set of states is $d_1 \cdots d_m - 1$ and the dimension of the set of product states is $d_1 + ... + d_m - m + 1$.
- Natural questions:
- How does one tell if a state is entangled?

- A product state is a state that can be represented as a tensor product $\phi_1 \otimes \cdots \otimes \phi_m$ with ϕ_i a state in $\mathcal{H}_i$.
- The simplest algebraic definition of an entangled state is a state that is not a product state.
- If $m > 1$ then a randomly chosen state will be entangled Since the dimension of the set of states is $d_1 \cdots d_m - 1$ and the dimension of the set of product states is $d_1 + ... + d_m - m + 1$.
- Natural questions:
- How does one tell if a state is entangled?
- Are there natural levels of entanglement?

- We look at EPR from the perspective of entanglement. We denote spin down by $|0\rangle$ and spin up by $|1\rangle$. The Hilbert space of spins of the electron is $\mathbb{C}^2$ with orthonormal basis $|0\rangle$, $|1\rangle$. That is qubits.

- We look at EPR from the perspective of entanglement. We denote spin down by $|0\rangle$ and spin up by $|1\rangle$. The Hilbert space of spins of the electron is $\mathbb{C}^2$ with orthonormal basis $|0\rangle, |1\rangle$. That is qubits.
- We can therefore look at the state of the pair of EPR electrons as

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

- We look at EPR from the perspective of entanglement. We denote spin down by $|0\rangle$ and spin up by $|1\rangle$. The Hilbert space of spins of the electron is $\mathbb{C}^2$ with orthonormal basis $|0\rangle$, $|1\rangle$. That is qubits.
- We can therefore look at the state of the pair of EPR electrons as

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

- The pair of electrons are in state $|0\rangle \otimes |1\rangle$ with probability $\frac{1}{2}$ and in state $|1\rangle \otimes |0\rangle$ with probability $\frac{1}{2}$. This says that if we measure the spin of the first electron and get spin up the state of spins of the electrons collapses to $|1\rangle \otimes |0\rangle$ and a measurement of the second electron is spin down.

- We look at EPR from the perspective of entanglement. We denote spin down by $|0\rangle$ and spin up by $|1\rangle$. The Hilbert space of spins of the electron is $\mathbb{C}^2$ with orthonormal basis $|0\rangle$, $|1\rangle$. That is qubits.
- We can therefore look at the state of the pair of EPR electrons as

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle).$$

- The pair of electrons are in state $|0\rangle \otimes |1\rangle$ with probability $\frac{1}{2}$ and in state $|1\rangle \otimes |0\rangle$ with probability $\frac{1}{2}$. This says that if we measure the spin of the first electron and get spin up the state of spins of the electrons collapses to $|1\rangle \otimes |0\rangle$ and a measurement of the second electron is spin down.
- Thus the state of the two electrons is entangled by the vague definition.

SLOCC

- SLOCC stands for stochastic local operations and classical communication.

SLOCC

- SLOCC stands for stochastic local operations and classical communication.
- Invertible SLOCC is just the action of $GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_m)$ on $\mathbb{P}(\mathcal{H})$. This is the same as the action of $SL(\mathcal{H}_1) \times \cdots \times SL(\mathcal{H}_m)$. We will denote it by SL.

SLOCC

- SLOCC stands for stochastic local operations and classical communication.
- Invertible SLOCC is just the action of $GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_m)$ on $\mathbb{P}(\mathcal{H})$. This is the same as the action of $SL(\mathcal{H}_1) \times \cdots \times SL(\mathcal{H}_m)$. We will denote it by SL.
- We will consider SL to be the product of the $SL(\mathcal{H}_i)$ in this lecture.

SLOCC

- SLOCC stands for stochastic local operations and classical communication.
- Invertible SLOCC is just the action of $GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_m)$ on $\mathbb{P}(\mathcal{H})$. This is the same as the action of $SL(\mathcal{H}_1) \times \cdots \times SL(\mathcal{H}_m)$. We will denote it by SL.
- We will consider SL to be the product of the $SL(\mathcal{H}_i)$ in this lecture.
- We note that the product states form a single orbit under the action of SL. So an SL invariant function on $\mathcal{H}$ is constant on the product states.

SLOCC

- SLOCC stands for stochastic local operations and classical communication.
- Invertible SLOCC is just the action of $GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_m)$ on $\mathbb{P}(\mathcal{H})$. This is the same as the action of $SL(\mathcal{H}_1) \times \cdots \times SL(\mathcal{H}_m)$. We will denote it by SL.
- We will consider SL to be the product of the $SL(\mathcal{H}_i)$ in this lecture.
- We note that the product states form a single orbit under the action of SL. So an SL invariant function on $\mathcal{H}$ is constant on the product states.
- Observe that the product states form the image, $\mathcal{U}$, of the Segre imbedding of $\mathbb{P}(\mathcal{H}_1) \times \cdots \times \mathbb{P}(\mathcal{H}_m)$ in $\mathbb{P}(\mathcal{H})$.

- If $I_{\mathcal{U}}$ is the homogeneous ideal of $\mathcal{U}$ in in the polynomials on $\mathcal{H}$ and if f is homogeneous function in $I_{\mathcal{U}}$ then $f(\mathcal{U}) = 0$. So if one such function on state v is non-zero the state is entangled.

- If $I_{\mathcal{U}}$ is the homogeneous ideal of $\mathcal{U}$ in in the polynomials on $\mathcal{H}$ and if f is homogeneous function in $I_{\mathcal{U}}$ then $f(\mathcal{U}) = 0$. So if one such function on state v is non-zero the state is entangled.
- Returning to two qubits. We note that if

$$v = \sum a_{ij} \left| ij \right\rangle = a_{00} \left| 00 \right\rangle + a_{01} \left| 01 \right\rangle + a_{10} \left| 10 \right\rangle + a_{11} \left| 11 \right\rangle .$$

Then

$$f(v) = \det \left[a_{ij} \right]$$

has the property that if v is a product state then $f(v) = 0$.

- If $I_{\mathcal{U}}$ is the homogeneous ideal of $\mathcal{U}$ in in the polynomials on $\mathcal{H}$ and if f is homogeneous function in $I_{\mathcal{U}}$ then $f(\mathcal{U}) = 0$. So if one such function on state v is non-zero the state is entangled.
- Returning to two qubits. We note that if

$$v = \sum a_{ij} |ij\rangle = a_{00} |00\rangle + a_{01} |01\rangle + a_{10} |10\rangle + a_{11} |11\rangle .$$

Then

$$f(v) = \det [a_{ij}]$$

has the property that if v is a product state then $f(v) = 0$.

- Also $f(EPR) = \frac{1}{2}$. So it is entangled in the algebraic sense.

- If $I_{\mathcal{U}}$ is the homogeneous ideal of $\mathcal{U}$ in in the polynomials on $\mathcal{H}$ and if f is homogeneous function in $I_{\mathcal{U}}$ then $f(\mathcal{U}) = 0$. So if one such function on state v is non-zero the state is entangled.
- Returning to two qubits. We note that if

$$v = \sum a_{ij} |ij\rangle = a_{00} |00\rangle + a_{01} |01\rangle + a_{10} |10\rangle + a_{11} |11\rangle .$$

Then

$$f(v) = \det [a_{ij}]$$

has the property that if v is a product state then $f(v) = 0$.

- Also $f(EPR) = \frac{1}{2}$. So it is entangled in the algebraic sense.
- One checks that in the case of qubits $I_{\mathcal{U}}$ is the ideal generated by f. Furthermore f is invariant under changes by SL.

- It is reasonable to think that a state (as an element of norm one) v that maximizes $|f|$ should be considered maximally entangled.

- It is reasonable to think that a state (as an element of norm one) v that maximizes $|f|$ should be considered maximally entangled.
- It is an exercise to show that the maximum of $|f|$ on sphere is $\frac{1}{2}$.

- It is reasonable to think that a state (as an element of norm one) v that maximizes $|f|$ should be considered maximally entangled.
- It is an exercise to show that the maximum of $|f|$ on sphere is $\frac{1}{2}$.
- Although this is over–kill, the Kemph-Ness theory implies that the states with $|f(v)| = \frac{1}{2}$ are precisely the elements of the orbit $(U(2) \otimes U(2))EPR$.

- It is reasonable to think that a state (as an element of norm one) v that maximizes $|f|$ should be considered maximally entangled.
- It is an exercise to show that the maximum of $|f|$ on sphere is $\frac{1}{2}$.
- Although this is over–kill, the Kemph-Ness theory implies that the states with $|f(v)| = \frac{1}{2}$ are precisely the elements of the orbit $(U(2) \otimes U(2))EPR$.
- We will come back to Kemph-Ness. But first what about entanglement in 3 qubits?

- By three qubits we mean $\mathcal{H}^3 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The action of SL is by outer tensor product action. If $v \in \mathcal{H}_3$ then we can write

$$v = |0\rangle \otimes v_0 + |1\rangle \otimes v_1$$

with $v_0, v_1 \in \mathcal{H}^2$.

- By three qubits we mean $\mathcal{H}^3 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The action of SL is by outer tensor product action. If $v \in \mathcal{H}_3$ then we can write

$$v = |0\rangle \otimes v_0 + |1\rangle \otimes v_1$$

  with $v_0, v_1 \in \mathcal{H}^2$.

- Polarizing f on $\mathcal{H}^2$ we have a non-degenerate SL invariant, symmetric, bilinear form on $\mathcal{H}^2$ which we denote by $(\ldots, \ldots)$. This leads to the tangle

$$\varphi(v) = \det \begin{bmatrix} (v_0, v_0) & (v_0, v_1) \\ (v_1, v_0) & v_1, v_1) \end{bmatrix}.$$

  An SL invariant homogeneous of degree 4 and known classically as the hyperdeterminant of a $2 \times 2 \times 2$ matrix.

- By three qubits we mean $\mathcal{H}^3 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The action of SL is by outer tensor product action. If $v \in \mathcal{H}_3$ then we can write

$$v = |0\rangle \otimes v_0 + |1\rangle \otimes v_1$$

  with $v_0, v_1 \in \mathcal{H}^2$.
- Polarizing f on $\mathcal{H}^2$ we have a non-degenerate SL invariant, symmetric, bilinear form on $\mathcal{H}^2$ which we denote by $(...,...)$. This leads to the tangle

$$\varphi(v) = \det \begin{bmatrix} (v_0, v_0) & (v_0, v_1) \\ (v_1, v_0) & v_1, v_1) \end{bmatrix}.$$

  An SL invariant homogeneous of degree 4 and known classically as the hyperdeterminant of a $2 \times 2 \times 2$ matrix.
- If $v = v_1 \otimes v_2 \otimes v_3$ with $v_i \neq 0$. Then there exists an element of SL that will transform v to $|0\rangle \otimes |0\rangle \otimes |0\rangle = |000\rangle$ so $\varphi(v) = 0$.

- The state studied by Greenberger, Horne and Zielinger:

$$GHZ = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

has $v_0 = \frac{1}{\sqrt{2}}|00\rangle$, $v_1 = \frac{1}{\sqrt{2}}|11\rangle$. Thus

$$\varphi(GHZ) = \det\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = -\frac{1}{4}.$$

- The state studied by Greenberger, Horne and Zielinger:

$$GHZ = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

has $v_0 = \frac{1}{\sqrt{2}}|00\rangle$, $v_1 = \frac{1}{\sqrt{2}}|11\rangle$. Thus

$$\varphi(GHZ) = \det\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = -\frac{1}{4}.$$

- Thus GHZ is entangled. We note that

$$\max_{\|v\|=1} |\varphi(v)| = \frac{1}{4}.$$

- The state studied by Greenberger, Horne and Zielinger:

$$GHZ = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

has $v_0 = \frac{1}{\sqrt{2}}|00\rangle$, $v_1 = \frac{1}{\sqrt{2}}|11\rangle$. Thus

$$\varphi(GHZ) = \det\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = -\frac{1}{4}.$$

- Thus GHZ is entangled. We note that

$$\max_{\|v\|=1} |\varphi(v)| = \frac{1}{4}.$$

- Furthermore, if $[GHZ]$ is the class of GHZ in $\mathbb{P}(\mathcal{H}^3)$ then the set $SL\cdot[GHZ]$ is open and dense in $\mathbb{P}(\mathcal{H}_3)$. Implying that the algebra of SL invariants is generated by φ. Kempf-Ness implies that $\frac{1}{4}$ is the maximum value and that the set of $v \in \mathcal{H}_3$, $\|v\| = 1$ with $|\varphi(v)| = \frac{1}{4}$ is exactly $(U(2) \otimes U(2) \otimes U(2))\, GHZ$.

- If we consider the hypersurface $\varphi = 0$ the there is a state

$$W = \frac{1}{\sqrt{3}} \left(|001\rangle + |010\rangle + |100\rangle \right)$$

with $SL \cdot W$ open in the hypersurface.

- If we consider the hypersurface $\varphi = 0$ the there is a state

$$W = \frac{1}{\sqrt{3}} \left(|001\rangle + |010\rangle + |100\rangle \right)$$

with $SL \cdot W$ open in the hypersurface.
- This state is entangled and many physicists list it and GHZ as equally entangled.

The Kempf-Ness Theorem

Let G be a connected reductive algebraic subgroup of $GL(\mathcal{H})$ (This implies that we may and do assume that if $g \in G$ then so is $g^* = g^\dagger$.) Set $K = U(\mathcal{H}) \cap G$. We say that $v \in \mathcal{H}$ is *critical* if $\langle Xv|v\rangle = 0$ for all $X \in Lie(G)$. We note that the set of critical points is invariant under the action of K. Here is the theorem.

Theorem

Let G, K be as above. Let $v \in \mathcal{H}$.
1. v is critical if and only if $\|gv\| \geq \|v\|$ for all $g \in G$.
2. If v is critical and $w \in Gv$ is such that $\|v\| = \|w\|$ then $w \in Kv$.
3. If Gv is closed then there exists a critical element in Gv.
4. If v is critical then Gv is closed.

- If $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ and $G = SL(\mathcal{H}_1) \otimes \cdots \otimes SL(\mathcal{H}_m)$ then the hypotheses of the above theorem are satisfied.

- If $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ and $G = SL(\mathcal{H}_1) \otimes \cdots \otimes SL(\mathcal{H}_m)$ then the hypotheses of the above theorem are satisfied.
- If ϕ is a homogeneous SL–invariant of degree $d > 0$ and if v is a state with $\phi(v) \neq 0$ then

$$\phi(\frac{gv}{\|gv\|}) = \|gv\|^{-d}\phi(gv) = \|gv\|^{-d}\phi(v).$$

Thus v maximizes $|\phi|$ on the set

$$\left\{ \frac{gv}{\|gv\|} | g \in G \right\}$$

if and only if it minimizes $\|gv\|$ for $g \in G$.

- If $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ and $G = SL(\mathcal{H}_1) \otimes \cdots \otimes SL(\mathcal{H}_m)$ then the hypotheses of the above theorem are satisfied.
- If ϕ is a homogeneous SL–invariant of degree $d > 0$ and if v is a state with $\phi(v) \neq 0$ then

$$\phi(\frac{gv}{\|gv\|}) = \|gv\|^{-d} \phi(gv) = \|gv\|^{-d} \phi(v).$$

  Thus v maximizes $|\phi|$ on the set

$$\left\{ \frac{gv}{\|gv\|} | g \in G \right\}$$

  if and only if it minimizes $\|gv\|$ for $g \in G$.
- For 2 and 3 qubits respectively and $v = EPR$ or GHZ we have $\left\{ \frac{gv}{\|gv\|} | g \in G \right\} = \{u \in \mathcal{H} | \, \|u\| = 1, h(u) \neq 0\}$, $h(u) = (u, u)$ or h is the tangle.

- If v is a state in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ the j–th factor of dimension d_j then we can expand v in terms of an orthonormal basis of $\mathcal{H}_1$, $|0\rangle, |1\rangle, ..., |d_j - 1\rangle$ as

$$\sum_{j=0}^{d_1-1} |j\rangle \otimes v_{1j}.$$

- If v is a state in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ the j–th factor of dimension d_j then we can expand v in terms of an orthonormal basis of $\mathcal{H}_1$, $|0\rangle, |1\rangle, ..., |d_j - 1\rangle$ as
$$\sum_{j=0}^{d_1-1} |j\rangle \otimes v_{1j}.$$
- If we permute the factors we have a similar expansion fro each factor. Thus getting v_{ij} for $i = 1, ..., m$ and $j = 0, ..., d_i - 1$.

- If v is a state in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ the j–th factor of dimension d_j then we can expand v in terms of an orthonormal basis of $\mathcal{H}_1$, $|0\rangle, |1\rangle, ..., |d_j - 1\rangle$ as

$$\sum_{j=0}^{d_1-1} |j\rangle \otimes v_{1j}.$$

- If we permute the factors we have a similar expansion fro each factor. Thus getting v_{ij} for $i = 1, ..., m$ and $j = 0, ..., d_i - 1$.
- v is critical if and only if

$$\langle v_{ik} | v_{il} \rangle = \frac{\delta_{kl}}{d_i}.$$

- If v is a state in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$ the j–th factor of dimension d_j then we can expand v in terms of an orthonormal basis of $\mathcal{H}_1$, $|0\rangle, |1\rangle, ..., |d_j - 1\rangle$ as
$$\sum_{j=0}^{d_1-1} |j\rangle \otimes v_{1j}.$$
- If we permute the factors we have a similar expansion fro each factor. Thus getting v_{ij} for $i = 1, ..., m$ and $j = 0, ..., d_i - 1$.
- v is critical if and only if
$$\langle v_{ik} | v_{il} \rangle = \frac{\delta_{kl}}{d_i}.$$
- This means that if I consider v to be a bipartite state by permuting the i–th factor to be first and thinking of the state as an element of
$$\mathcal{H}_i \otimes \left(\mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m \right)$$
The reduced trace of v is $\frac{1}{d_i} I$.

- We can rephrase this as follows: If v is a state then for each i there is an operator $T_{i,v} : \mathcal{H}_i \longrightarrow \mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m$ that sends $|j\rangle$ to v_{ij}. The reduced trace is the operator $T_i^\dagger T_i$.

- We can rephrase this as follows: If v is a state then for each i there is an operator $T_{i,v} : \mathcal{H}_i \to \mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m$ that sends $|j\rangle$ to v_{ij}. The reduced trace is the operator $T_i^\dagger T_i$.
- The von Neumann Entropy of the mixed state (i.e. a non-negative operator of trace 1), $A = T_i^\dagger T$, is

$$-\mathrm{tr} A \log A.$$

- We can rephrase this as follows: If v is a state then for each i there is an operator $T_{i,v} : \mathcal{H}_i \rightarrow \mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m$ that sends $|j\rangle$ to v_{ij}. The reduced trace is the operator $T_i^{\dagger} T_i$.
- The von Neumann Entropy of the mixed state (i.e. a non-negative operator of trace 1), $A = T_i^{\dagger} T$, is

$$-\mathrm{tr} A \log A.$$

- v is critical if and only if $T_i^{\dagger} T_i = \frac{1}{d_i} I$. Thus Kempf-Ness implies

- We can rephrase this as follows: If v is a state then for each i there is an operator $T_{i,v} : \mathcal{H}_i \rightarrow \mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m$ that sends $|j\rangle$ to v_{ij}. The reduced trace is the operator $T_i^\dagger T_i$.
- The von Neumann Entropy of the mixed state (i.e. a non-negative operator of trace 1), $A = T_i^\dagger T$, is

$$-\mathrm{tr} A \log A.$$

- v is critical if and only if $T_i^\dagger T_i = \frac{1}{d_i} I$. Thus Kempf-Ness implies
- The von Neumann entropy of each mixed state gotten by tracing out all factors but one of v for all choiced is maximal if and only if it is critical.

- We can rephrase this as follows: If v is a state then for each i there is an operator $T_{i,v} : \mathcal{H}_i \rightarrow \mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m$ that sends $|j\rangle$ to v_{ij}. The reduced trace is the operator $T_i^\dagger T_i$.
- The von Neumann Entropy of the mixed state (i.e. a non-negative operator of trace 1), $A = T_i^\dagger T$, is

$$-\mathrm{tr} A \log A.$$

- v is critical if and only if $T_i^\dagger T_i = \frac{1}{d_i} I$. Thus Kempf-Ness implies
- The von Neumann entropy of each mixed state gotten by tracing out all factors but one of v for all choiced is maximal if and only if it is critical.
- If there exists a closed non-zero SL orbit then $2 \dim H_i \leq \dim \mathcal{H}$ for all i.

- We can rephrase this as follows: If v is a state then for each i there is an operator $T_{i,v} : \mathcal{H}_i \to \mathcal{H}_1 \otimes \cdots \otimes \widehat{\mathcal{H}_i} \otimes \cdots \otimes \mathcal{H}_m$ that sends $|j\rangle$ to v_{ij}. The reduced trace is the operator $T_i^\dagger T_i$.
- The von Neumann Entropy of the mixed state (i.e. a non-negative operator of trace 1), $A = T_i^\dagger T$, is

$$-\mathrm{tr} A \log A.$$

- v is critical if and only if $T_i^\dagger T_i = \frac{1}{d_i} I$. Thus Kempf-Ness implies
- The von Neumann entropy of each mixed state gotten by tracing out all factors but one of v for all choiced is maximal if and only if it is critical.
- If there exists a closed non-zero SL orbit then $2 \dim H_i \leq \dim \mathcal{H}$ for all i.
- This is precisely the condition for the existence of the $d_1 \times d_2 \times \cdots \times d_m$ hyperdeterminant.

Another basic result in the theory is the extended Hilbert-Mumford theorem (which is used in the proof of the hard part of the Kempf-Ness Theorem). Let G, $\mathcal{H}$ be as in the Kempf-Ness theorem. Then one knows that a G orbit contains a unique closed orbit in its closure.

Theorem

Let $v \in \mathcal{H}$ and let Gw be the closed orbit in $\overline{Gv}$. Then there exists an algebraic group homomorphism $\varphi : \mathbb{C}^\times \to G$ such that $\lim_{z \to 0} \varphi(z)v \in Gw$. Furthermore φ can be chosen so that $\varphi(\overline{z}) = \varphi(z)^*$.

- We also note that in the context of the theorems the set of zeros of homogeneous polynomial invariants of positive degree is called the null cone. The only closed orbit in the null cone is $\{0\}$.

- We also note that in the context of the theorems the set of zeros of homogeneous polynomial invariants of positive degree is called the null cone. The only closed orbit in the null cone is $\{0\}$.
- For 3 qubits the invariants are polynomials in the tangle so the W state is in the null cone. Thus the Hilbert-Mumford theorem implies the existence of $\varphi : \mathbb{C}^{\times} \longrightarrow G$ such that $\lim_{z\to 0} \varphi(z)W = 0$.

- We also note that in the context of the theorems the set of zeros of homogeneous polynomial invariants of positive degree is called the null cone. The only closed orbit in the null cone is $\{0\}$.
- For 3 qubits the invariants are polynomials in the tangle so the W state is in the null cone. Thus the Hilbert-Mumford theorem implies the existence of $\varphi : \mathbb{C}^{\times} \to G$ such that $\lim_{z\to 0} \varphi(z)W = 0$.
- Relative to the reduced trace we can choose any k of the factors of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$, $\mathbf{i} = i_1 < \cdots < i_k$ and consider $\mathbf{j} = j_1 < \cdots < j_{m-k}$ the complementary indices and get a linear map

$$T_{\mathbf{i},\mathbf{j}} : \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_k} \to \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_{m-k}}.$$

- We also note that in the context of the theorems the set of zeros of homogeneous polynomial invariants of positive degree is called the null cone. The only closed orbit in the null cone is $\{0\}$.
- For 3 qubits the invariants are polynomials in the tangle so the W state is in the null cone. Thus the Hilbert-Mumford theorem implies the existence of $\varphi : \mathbb{C}^\times \to G$ such that $\lim_{z\to 0} \varphi(z)W = 0$.
- Relative to the reduced trace we can choose any k of the factors of $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m$, $\mathbf{i} = i_1 < \cdots < i_k$ and consider $\mathbf{j} = j_1 < \cdots < j_{m-k}$ the complementary indices and get a linear map

$$T_{\mathbf{i},\mathbf{j}} : \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_k} \to \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_{m-k}}.$$

- Set $d_{\mathbf{i}} = \dim \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_k}$ then assuming $d_{\mathbf{i}} \leq d_{\mathbf{j}}$ we can ask does there exist a state such that $T_{\mathbf{i},\mathbf{j}}^\dagger T_{\mathbf{i},\mathbf{j}} = \frac{1}{d_{\mathbf{i}}} I$ for all such choices of $\mathbf{i}$?

- Arguably such a state should be considered maximally entangled.

- Arguably such a state should be considered maximally entangled.
- For 2 qubits this condition describes the orbit of EPR.up to local unitary transformations. For 3 qubits it describes GHZ up to local unitary transformations.

- Arguably such a state should be considered maximally entangled.
- For 2 qubits this condition describes the orbit of EPR.up to local unitary transformations. For 3 qubits it describes GHZ up to local unitary transformations.
- Rains has shown that if $|0\rangle \longmapsto u, |1\rangle \longmapsto v$ defines the perfect 5 qubit error correcting code then the orbit of u (or or v) is described by this condition. For 6 qubits we have

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes u + |1\rangle \otimes v\right).$$

- Arguably such a state should be considered maximally entangled.
- For 2 qubits this condition describes the orbit of EPR.up to local unitary transformations. For 3 qubits it describes GHZ up to local unitary transformations.
- Rains has shown that if $|0\rangle \longmapsto u, |1\rangle \longmapsto v$ defines the perfect 5 qubit error correcting code then the orbit of u (or or v) is described by this condition. For 6 qubits we have

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes u + |1\rangle \otimes v\right).$$

- For 4 qubits no such state exists. One can show that there are 90 orbit types using Kostant-Rallis theory for the group D_4.

- The Bell states

$$u_0 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), u_1 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$

$$u_2 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle), u_3 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle).$$

All in the orbit of *EPR* (the last one) leads to a basis for the space of 4 qubits $u_i \otimes u_j$.

- The Bell states
$$u_0 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), u_1 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$
$$u_2 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle), u_3 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle).$$
All in the orbit of EPR (the last one) leads to a basis for the space of 4 qubits $u_i \otimes u_j$.
- The 4 dimensional space A with basis $v_j = u_j \otimes u_j$ is particularly interesting.

- The Bell states

$$u_0 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), u_1 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$

$$u_2 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle), u_3 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle).$$

  All in the orbit of EPR (the last one) leads to a basis for the space of 4 qubits $u_i \otimes u_j$.
- The 4 dimensional space A with basis $v_j = u_j \otimes u_j$ is particularly interesting.
- The polynomial invariants under SL are generated by 4 invariants, f_2, f_4, ph, f_6, whose restriction to the elements $\sum x_i v_i$ are given as

$$\sum x_i^2, \sum x_i^4, x_0 x_1 x_2 x_3, \sum x_i^6.$$

- The Bell states
$$u_0 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), u_1 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$
$$u_2 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle), u_3 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle).$$
All in the orbit of EPR (the last one) leads to a basis for the space of 4 qubits $u_i \otimes u_j$.
- The 4 dimensional space A with basis $v_j = u_j \otimes u_j$ is particularly interesting.
- The polynomial invariants under SL are generated by 4 invariants, f_2, f_4, ph, f_6, whose restriction to the elements $\sum x_i v_i$ are given as
$$\sum x_i^2, \sum x_i^4, x_0 x_1 x_2 x_3, \sum x_i^6.$$
- The critical elements consist of the set $(U(2) \otimes U(2) \otimes U(2) \otimes U(2))A$.

- The Bell states

$$u_0 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), u_1 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle),$$

$$u_2 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle), u_3 = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle).$$

  All in the orbit of EPR (the last one) leads to a basis for the space of 4 qubits $u_i \otimes u_j$.
- The 4 dimensional space A with basis $v_j = u_j \otimes u_j$ is particularly interesting.
- The polynomial invariants under SL are generated by 4 invariants, f_2, f_4, ph, f_6, whose restriction to the elements $\sum x_i v_i$ are given as

$$\sum x_i^2, \sum x_i^4, x_0x_1x_2x_3, \sum x_i^6.$$

- The critical elements consist of the set $(U(2) \otimes U(2) \otimes U(2) \otimes U(2))A$.
- A complete discussion of entanglement for 4 qubits can be found in my paper with Gilad Gour, "All maximally entangled 4 qubit states" most easily found on the archive.

Among the 9 closed orbit types 2 examples stand out.

$$L = \frac{1}{\sqrt{3}} \left(v_0 + \omega v_1 + \omega^2 v_2 \right), \omega = e^{\frac{2\pi i}{3}},$$

$$M = \frac{i}{\sqrt{2}} v_0 + \frac{1}{\sqrt{6}} \left(v_1 + v_2 + v_3 \right).$$

The various total entropies that depend on parameters have values running between the value at each of these. That both maximize total Von Neumann 2,2 entropy and considering other total 2,2 entropies of states with maximal Von Neuman 2,2 entropy one is maximal and the other is minimal.