A Tale of Four Simple Groups over the Reals

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The quaternionic groups

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- Fix a maximal torus, T, of U and let $\mathfrak{h}$ denote its complexified Lie algebra. Let Φ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and let Φ^+ be a choice of positive roots. If $\alpha \in \Phi$ let $\check{\alpha} \in \mathfrak{h}$ denote the corresponding coroot. Let τ denote the complex conjugation on $G_{\mathbb{C}}$ with respect to the real form U. We set $\sigma = Ad(\exp(\pi i \check{\alpha}_o))\tau$ where α_o is the highest root with respect to the choice of Φ^+.

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- Then σ is a complex conjugation on $G_{\mathbb{C}}$. The fixed point set of σ is up to conjugacy the quaternionic real form of $G_{\mathbb{C}}$ which we will denote by G. The Cartan involution that corresponds to the maximal compact subgroup $K = G \cap U$ is $\theta = Ad(\exp(\pi i \check{\alpha}_o))$.

- The real form is called quaternionic since the subgroup of G corresponding to α_o is isomorphic to $SU(2)$ (the unit quaternions) and the action of this $SU(2)$on the negative one eigenspace of θ in $\mathfrak{g}$ is an even multiple of the two dimensional representation.

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- This implies that K contains a normal subgroup, $K_{o,}$ isomorphic with $SU(2)$and another normal subgroup K_1 of codimension 3 such that $K = K_o \cdot K_1$. Let T_o be a maximal torus in K_o.We set $L = T_o K_1 = C_G(T_o)$.

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- If $\mathfrak{q}$ is the parabolic subalgebra of $\mathfrak{g}$ that is given by the sum of the non-negative eigenspaces of $ad(\check{\alpha}_o)$ and if Q is the normalizer of $\mathfrak{q}$ in $G_{\mathbb{C}}$ then the complex structure comes from $G/L = G_{\mathbb{C}}/Q$.

- In our 1996 paper Dick Gross and I consider the holomorphic line bundles, $\mathcal{L}_\lambda$, over G/L corresponding to unitary characters, λ, of L trivial on K_1. These characters have differential $k\frac{\alpha_o}{2}$. Using methods of Schmid's thesis we show that if $k \geq 2$ then the only non-zero sheaf cohomology is in degree 1.

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- Under this condition we show that the $(\mathfrak{g}, K)$ module of K-finite cohomology $H^1(G/L, \mathcal{L}_\lambda)_K$ has a unique irreducible submodule. For each of the exceptional groups of real rank 4 this yields 3 cases where the subrepresentation is proper (for D_4 we will give two).

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- We will, at first, restrict our attention to the exceptional quaternionic real forms and $S_3 \ltimes SO(4,4)_o$.
- $Q = L_\mathbb{C} U$ with U the unipotent radical of Q. We set $V = Lie(U)/[Lie(U), Lie(U)]$. V is a symplectic vector space since U is a Heisenberg group. The three representations above follow the orbit structure of the action of $L_\mathbb{C}$ on $\mathbb{P}(V)$.

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3. In X_2 there is one open orbit, $\mathcal{O}_3$. In the case of D_4 this orbit has three components permuted by the S_3.
4. The complement of $\mathcal{O}_3$in X_2 is the closed orbit $\mathcal{O}_4 = X_3$.

- The point here is that the K-spectrum of each of these representations is of the form

$$\bigoplus_{n\geq 0} S^{k-2+n}(\mathbb{C}^2)\otimes A^n(Y).$$

Here Y is an $L_{\mathbb{C}}$ invariant closed subvariety of $\mathbb{P}(V)$, $A^n(Y)$ is the space of degree n elements of the homogeneous coordinate ring. Here is the table of values of k and and Y.

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D_4	F_4	E_6	E_7	E_8	Y
4	7	10	16	28	X_1
	4	6	10	18	X_2
2	3	4	6	10	X_3

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- The point here is that if $f = 0,1,2,4,8$ for D_4, F_4, E_6, E_7 and E_8 respectively then the numbers appearing are $3f+4, 2f+2, f+2$. We will now look at the next level but only for the exceptional groups the meaning of the numbers f will be more apparent.

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- For all of the other groups the last row yields their minimal representation. The results in all cases are analogous to my results for holomorphic representations.

The next level of groups

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- Let $\mathfrak{g}_F$ be the real vector space consisting of the $2n \times 2n$ matrices of with block form
$$\begin{bmatrix} A & X \\ Y & -A^* \end{bmatrix}$$
with $A, X, Y \in M_n(F)$ and $X^* = X, Y^* = Y$. An easy check shows that $\mathfrak{g}_F$ is a Lie subalgebra of $M_{2n}(F)$.

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- The group K is respectively locally $U(n), U(n) \times U(n)$ and $U(2n)$.

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- A reinterpretation of the block form above sets up the new example. First we note that we can look upon $\mathcal{A}_F = \{X \in M_n(F) | X^* = X\}$ as a Jordan algebra under $X \circ Y = \frac{1}{2}(XY + YX)$. The automorphism groups of these Jordan algebras are $O(n)$, $U(n)$ and $Sp(n)$ under the obvious action. The upshot is that we can look upon the Cartan decomposition of $M_n(F)$ as giving a direct sum decomposition $Der(\mathcal{A}_F) \oplus \{L_X | X \in \mathcal{A}_F\}$. $L_X Y = X \circ Y$. The total Lie algebra is $\mathcal{A}_F^* \oplus (Der(\mathcal{A}_F) \oplus \{L_X | X \in \mathcal{A}_F\}) \oplus \mathcal{A}_F$.

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- $\mathcal{A}_{\mathbb{O}} = \{X \in M_3(\mathbb{O}) | X^* = X\}$ under $X \circ Y = \frac{1}{2}(XY + YX)$ forms a Jordan algebra. $Der(\mathcal{A}_{\mathbb{O}})$ is isomorphic with the compact real form of F_4. The Lie algebra $Der(\mathcal{A}_{\mathbb{O}}) \oplus \{L_X | X \in \mathcal{A}_{\mathbb{O}}\}$ is isomorphic to the direct sum of a one dimensional center and a rank 2 real form of E_6. The total Lie algebra (putting together all the parts) is the rank 3 real form of E_7.

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- We set $V_{\mathbb{R}} = Lie(N/[N,N])$. Then the generic unitary characters of $\overline{N}$ (identified with $V_{\mathbb{R}}$) for which the quaternionic discrete series has a generalized Whittaker model form one open orbit under MA. We note that there are 4 open orbits.

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- We now move to the groups M.

- For F_4, E_6, E_7, E_8, respectively, we assign the field of dimension $1, 2, 4$ or 8, F. Then for the first 3, $\not{M}A$ is the subgroup of $GL(6, F)$ corresponding to the Lie algebra constructed above for $n = 3$. For the octonions the group is the real form of E_7 constructed above. In other words these are groups of automorphisms of the Hermitian symmetric tube domains of rank 3.

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- Associated to M is a conjugacy class of real parabolic subgroups with abelian nilradical. Their Lie algebras can be described, in the notation above, as $(Der(\mathcal{A}_F) \oplus \{L_a | a \in \mathcal{A}_F\}) \oplus \mathcal{A}_F$.

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- This is the Shilov boundary parabolic for each of these tube domains. The full Lie algebra is

$$\mathcal{A}_F^* \oplus (Der(\mathcal{A}_F) \oplus \{L_a | a \in \mathcal{A}_F\}) \oplus \mathcal{A}_F.$$

- We note that if $\overline{N}$ is the unipotent radical of the corresponding opposite parabolic subgroup of M then the unitary characters of $\overline{N}$ are given by elements of $\mathcal{A}_F$. The element 1 has as stabilizer in M the compact symmetric subgroup corresponding to $Der(\mathcal{A}_F)$.

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- This condition allows one to characterize all Bessel models for admissible representations of these groups. This problem was solved this year.
- Set H equal to the normalizer in G of $Der(\mathcal{A}_F) \oplus \{L_a | a \in \mathcal{A}_F\}$. This is the next to the last level of groups we will study.

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- A maximal compact subgroup of H, K_H, is respectively $O(3), U(3), Sp(3)$ and F_4.
- The flag varieties are thus given by $K_H/B \cap K_H$.

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- Notice that each of these groups has triality (an S_3) of outer automorphisms.
- Using this triality, in my 1972, Annals paper I showed that each of these flag varieties admitted a, homogeneous, positively pinched, Riemannian structure.
- The only known examples of simply connected, compact, manifolds admitting a positively pinched Riemannian structure of dimension greater than 24 are the spheres and projective spaces over $F = \mathbb{C}, \mathbb{H}$.