6. [20 points] A patch $\mathbf{f}(\alpha, \beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 0, 0, 0\rangle$, $\mathbf{y} = \langle 6, 0, 3\rangle$, $\mathbf{z} = \langle 6, 6, 0\rangle$, and $\mathbf{w} = \langle 0, 6, 0\rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$.

(a) Give the parametric formula $\mathbf{q}(u, v)$ for the patch.

(b) What is the point on this patch with bilinear coordinates $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$?

(c) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

4. [15 points]

a. Give the definition of an "Affine Transformation" mapping $\mathbb{R}^n \to \mathbb{R}^n$.

b. Give the definition of an "Affine Combination" of k points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$.

4. [20 points] Briefly describe the three listed methods below. Make it clear how they differ.

(a) Supersampling,

(b) Stochastic Supersampling, and

(c) Jittered Stochastic Supersampling.

6. (Phong lighting — specular.) Describe how to compute the ***specular*** component of Phong lighting (but not including the Schlick-Fresnel component). Do this for a ***single*** light source and a ***single*** color.

- Describe all input values, both scalars and vectors. What are their meanings? Do the vectors need to be unit vectors?
- Draw a picture showing the surface, the light, the viewer and the relevant vectors.
- Give the complete algorithms and formulas needed for the reflection vector method and the halfway vector method for computing Phong lighting.

8. Let $\mathbf{x} = \langle 0,0,0\rangle$, $\mathbf{y} = \langle 5,0,1\rangle$, $\mathbf{z} = \langle 4,1,1\rangle$, and $\mathbf{w} = \langle -1,2,0\rangle$ be the four vertices of a quadrangle in counterclock-wise order. For each pair of values α and β, what point is obtained by bilinear interpolation in this quadrangle? (Or, if no such point exists, explain why not.)

a. $\alpha = 0$ and $\beta = 1$.

b. $\alpha = 1$ and $\beta = 1$.

c. $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$.

d. $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{3}$.

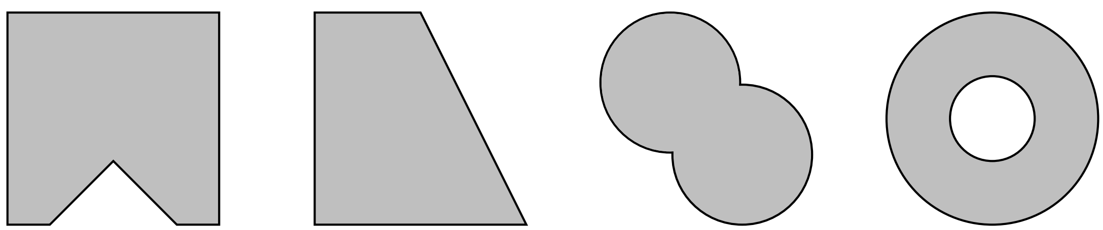
6. [20 points] A patch $\mathbf{f}(\alpha,\beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 0,0,0\rangle$, $\mathbf{y} = \langle 6,0,3\rangle$, $\mathbf{z} = \langle 6,6,0\rangle$, and $\mathbf{w} = \langle 0,6,0\rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$.

(a) Give the parametric formula $\mathbf{q}(u,v)$ for the patch.

(b) What is the point on this patch with bilinear coordinates $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$?

(c) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

8. [16 points] Which of the following four figures are convex? Which are not convex? (Write your answers under each figure.)



2. Hyperbolic interpolation.

This problem concerns interpolation in homogeneous coordinates or hyperbolic interpolation. Suppose that two 4-vectors $\mathbf{v}$ and $\mathbf{w}$ are equal to

$$\mathbf{v} = \langle 4,0,4,2\rangle \qquad \text{and} \qquad \mathbf{w} = \langle 0,0,0,1\rangle.$$

These are homogeneous coordinates for the points $\langle 2,0,2\rangle$ and $\langle 0,0,0\rangle$ in $\mathbb{R}^3$.

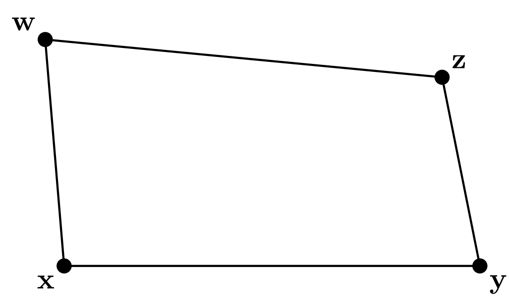
2.a. Find a value for α so that $\text{L}\textsc{erp}(\mathbf{v},\mathbf{w},\alpha)$ is a homogeneous representation of the midpoint $\langle 1,0,1\rangle$ in $\mathbb{R}^3$.

2.b. Write out $\text{L}\textsc{erp}(\mathbf{v},\mathbf{w},\alpha)$ explicitly (as a 4-vector).

2. [20 points] This problem concerns bilinear interpolation

$$\mathbf{u}(\alpha, \beta) \;=\; (1-\alpha)(1-\beta)\mathbf{x} + \alpha(1-\beta)\mathbf{y} + \alpha\beta\mathbf{z} + (1-\alpha)\beta\mathbf{w}$$

in $\mathbb{R}^3$. Let $\mathbf{x} = \langle -1, -1, 0\rangle$ and $\mathbf{y} = \langle 1, -1, 2\rangle$ and $\mathbf{z} = \langle 1, 1, 1\rangle$ and $\mathbf{w} = \langle -1, 1, 1\rangle$.



a. On the picture above draw the approximate locations of the following points and label them. (You do not need to calculate the points, just draw their approximate location.)

The point $\mathbf{s}$ with bilinear coordinates $\alpha = 0.9$ and $\beta = 0.1$.

The point $\mathbf{t}$ with bilinear coordinates $\alpha = 0$ and $\beta = 1$.

The point $\mathbf{u}$ with bilinear coordinates $\alpha = 0.5$ and $\beta = 0.5$.

b. The bilinear interpolation on $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{w}$ defines a (parametric) surface $\mathbf{q}(\alpha, \beta)$. Give a normal vector for this surface at the point $\mathbf{u}$ given above.

10. [20 points] A patch $\mathbf{p}(\alpha, \beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 1, 0, 1\rangle$, $\mathbf{y} = \langle 1, 2, -1\rangle$, $\mathbf{z} = \langle -1, 2, 1\rangle$, and $\mathbf{w} = \langle -1, 0, -1\rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$. (You may wish to work this problem on scratch paper first, and then transfer your work to the exam.)

(a) The point $\mathbf{v} = \langle -1, \frac{3}{2}, \frac{1}{2}\rangle$ lies on the line segment joining $\mathbf{z}$ and $\mathbf{w}$. What are the bilinear coordinates α and β for the point $\mathbf{v}$?

(b) What is the point $\mathbf{u}$ on this patch with bilinear coordinates $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$?

(c) Give the values of the partial derivatives at this point $\mathbf{u}$: $\frac{\partial \mathbf{p}}{\partial \alpha}(\frac{1}{4}, \frac{1}{2})$ and $\frac{\partial \mathbf{p}}{\partial \beta}(\frac{1}{4}, \frac{1}{2})$.

(d) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

10. [20 points] A patch $\mathbf{p}(\alpha,\beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 1,0,1\rangle$, $\mathbf{y} = \langle 1,2,-1\rangle$, $\mathbf{z} = \langle -1,2,1\rangle$, and $\mathbf{w} = \langle -1,0,-1\rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$. (You may wish to work this problem on scratch paper first, and then transfer your work to the exam.)

(a) The point $\mathbf{v} = \langle -1, \frac{3}{2}, \frac{1}{2}\rangle$ lies on the line segment joining $\mathbf{z}$ and $\mathbf{w}$. What are the bilinear coordinates α and β for the point $\mathbf{v}$?

(b) What is the point $\mathbf{u}$ on this patch with bilinear coordinates $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$?

(c) Give the values of the partial derivatives at this point $\mathbf{u}$: $\frac{\partial \mathbf{p}}{\partial \alpha}(\frac{1}{4},\frac{1}{2})$ and $\frac{\partial \mathbf{p}}{\partial \beta}(\frac{1}{4},\frac{1}{2})$.

(d) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

7. A piecewise degree Bézier curve is to be composed of two pieces, $\mathbf{q}_1(\mathbf{u})$ and $\mathbf{q}_2(\mathbf{u})$, both defined on the domain [0, 1]. The control points of $\mathbf{q}_1(u)$ are

$$\mathbf{p}_0 = \langle 0, 0\rangle \text{ and } \mathbf{p}_1 = \langle 0, 1\rangle \text{ and } \mathbf{p}_2 = \langle 1, 2\rangle \text{ and } \mathbf{p}_3 = \langle 2, 2\rangle.$$

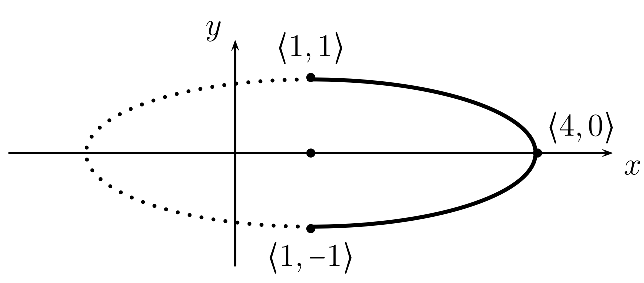
The second piece, $\mathbf{q}_2(u)$ is to be defined with control points $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. We want to have $\mathbf{q}_2(0) = \mathbf{q}_1(1)$ and to have $\mathbf{q}_2(1) = \langle 4, 0\rangle$.

a. What are the values $\mathbf{r}_0$ and $\mathbf{r}_3$?

b. Suppose we want the overall curve to be G^1-continuous and to have $\mathbf{q}_2'(1) = \langle 1,0\rangle$. Describe the set of all possible values for the remaining two control points $\mathbf{r}_1$ and $\mathbf{r}_2$ that make these conditions hold.

c. Now suppose we want the overall curve to be C^1-continuous and to have $\mathbf{q}_2'(1) = \mathbf{q}_2'(0)$. Describe the set of all possible values for the remaining two control points $\mathbf{r}_1$ and $\mathbf{r}_2$ that make these conditions hold.

3. This question concerns a rational Bézier curve in $\mathbb{R}^2$.
An ellipse in $\mathbb{R}^2$ is centered at $\langle 1,0\rangle$ and has major radius 3 and minor radius 1. Its major radius is along the x-axis; its minor radius is parallel to the y-axis. Thus it goes through the four points $\langle 1,\pm 1\rangle$, $\langle -2,0\rangle$ and $\langle 4,0\rangle$.



a. Express the right half of this ellipse as a degree 2 Bézier curve by giving its control points.

b. Now express the same curve as a degree 3 Bézier curve.