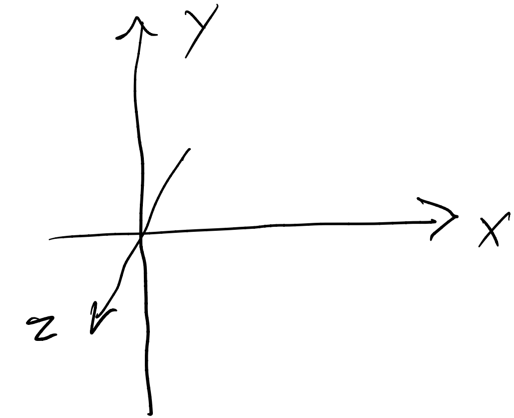
Moving to $\mathbb{R}^3$ - (3-space)

$$\vec{x} = \langle x_1, x_2, x_3 \rangle = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



z-axis towards the viewer.

$\vec{\imath} \times \vec{\jmath} = \vec{k}$ - obeys right hand rule for crossproducts

$\vec{u} \times \vec{v}$

Defn Linear transform - same definition as before.

Affine transformation - " " " "

$$A(\vec{x}) = B(\vec{x}) + \vec{u} \quad \text{where } B \text{ is linear.}$$

Translation $T_{\vec{u}}$ for $\vec{u} \in \mathbb{R}^3$

$$T_{\vec{u}}(\vec{x}) = \vec{x} + \vec{u}.$$

Example $\vec{u} = \langle 1, 0, 0 \rangle$. $T_{\vec{u}}(\langle x, y, z \rangle) = \langle x+1, y, z \rangle$

3×3 matrix representation of a linear map $A : \mathbb{R}^3 \to \mathbb{R}^2$

If $A(\vec{i}) = \vec{u}$ $A(\vec{j}) = \vec{v}$ $A(\vec{k}) = \vec{w}$

then A is represented by

$$(\vec{u}\ \vec{v}\ \vec{w}) = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

e.g. $M\vec{i} = M \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \vec{u}.$

Example: The rotation $R_{90^\circ, \vec{j}}$ or $R_{\pi/2, \vec{j}}$.

rotates around the vector $\vec{j}$ (y-axis)
90° in the counterclockwise direction
viewed from above (as given by the
righthand rule).

Question What 3×3 matrix represents $R_{\pi/2, \vec{j}}$?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Acting on homogeneous coordinates

we: $$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Homogeneous coordinates

<u>Refresher in $\mathbb{R}^2$</u>

$\langle x, y, w \rangle$ homogeneous coordinates for $\langle x/w, y/w \rangle$

$$A(\vec{x}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x} + \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{- affine}$$

The 3x3 matrix $\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}$

$\langle 3, 5, 1 \rangle$, $\langle 3/2, 5/2, 1/2 \rangle$, $\langle 6, 10, 2 \rangle$

all represent $\langle 3, 5 \rangle \in \mathbb{R}^2$

In $\mathbb{R}^3$, $\langle x, y, z, w\rangle$ homogeneous coordinates for $\langle x/w, y/w, z/w\rangle$

An affine map $A: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$A\vec{x} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix} \vec{x} + \begin{pmatrix} m \\ n \\ p \end{pmatrix} \quad \text{- affine}$$

represent by

$$\begin{pmatrix} a & b & c & m \\ d & e & f & n \\ g & h & l & p \\ 0 & 0 & 0 & 1 \end{pmatrix} =: M$$

$M\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ gives $\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix}$ where $A\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$.

<u>Example</u> Uniform Scaling $S_\alpha(\langle x, y, z\rangle) = \langle \alpha x, \alpha y, \alpha z\rangle$

Nonuniform scaling $S_{\langle \alpha, \beta, \gamma\rangle}(\langle x, y, z\rangle) = \langle \alpha x, \beta y, \gamma z\rangle$

α

β

γ γ

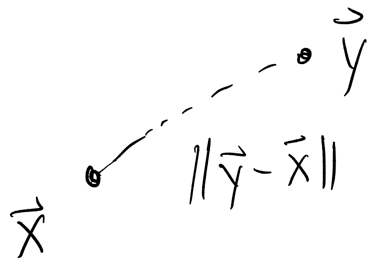
These are linear

$S_{\langle \alpha, \beta, \gamma\rangle}$ represented 3x3 matrix

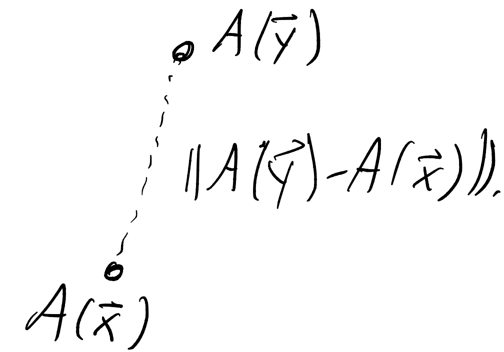
$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

$$S_{\alpha\beta\gamma}\begin{pmatrix} 1 \\ 9 \end{pmatrix} = \begin{pmatrix} \alpha \\ 9 \end{pmatrix}$$

Defn A transformation A is rigid if for all $\vec{x}, \vec{y}$, $\|A(\vec{x}) - A(\vec{y})\| = \|\vec{x} - \vec{y}\|$ – i.e., A preserves distances (between points).



$\|\vec{y} - \vec{x}\|$ - distance between $\vec{x}$ and $\vec{y}$



$\|\vec{y}\|$ - magnitude or norm or length of $\vec{y}$

Example A translation $T_{\vec{u}}$ or a rotation $R_{\theta,\vec{u}}$ are rigid.

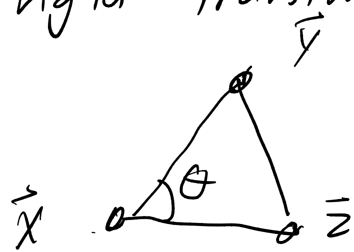
"Rigid" means sizes or shapes do not (except ......)!

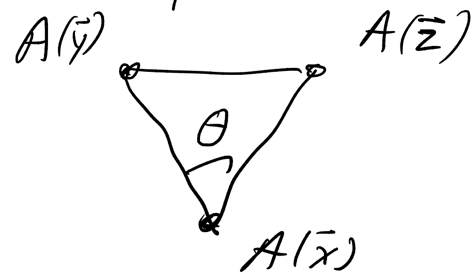
Also a reflection, example $S_{\langle -1,1,1\rangle}$

$$A(\langle x,y,z\rangle) = \langle -x, y, z\rangle$$

is also rigid. (So "exception" refers to reflections).

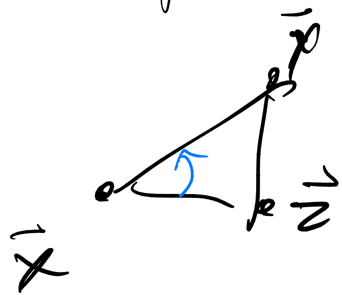
Observation: A rigid transformation also preserves angles

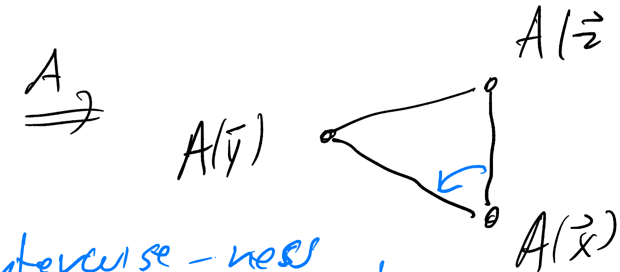




By SSS theorem (Side-Side-Side)

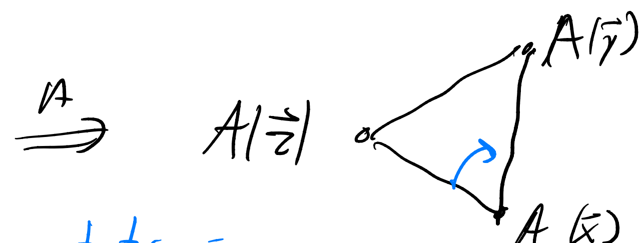
Def'n In $\mathbb{R}^2$, $A:\mathbb{R}^2 \to \mathbb{R}^2$ is orientation preserving if it preserves directions of angles





Counterwise-ness of angle is unchanged

— So is orientation preserving



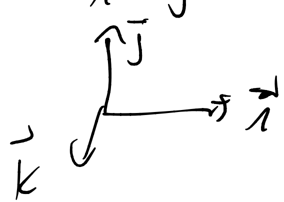
Not orientation-preserving.

In $\mathbb{R}^3$: A ~~linear~~ transformation $A: \mathbb{R}^3 \to \mathbb{R}^3$ is orientation preserving if it preserves the sign of triple products $(\vec{u} \times \vec{v}) \cdot \vec{w}$

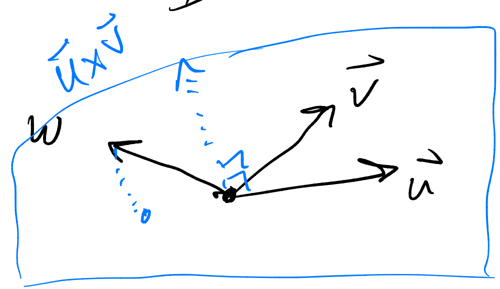
Iff: $\left(A(\vec{\imath}) \times A(\vec{\jmath})\right) \cdot A(\vec{k}) > 0$.

I.e. intuition that righthandness of orientations of triples of vectors

$\vec{\imath} \times \vec{\jmath} = \vec{k}$



If $\vec{u} = A(\vec{\imath})$, $\vec{v} = A(\vec{\jmath})$, $\vec{w} = A(\vec{k})$



Plane with $\vec{u}, \vec{v}$
$\vec{w}$ – "above" it

In $\mathbb{R}^3$, Translations $T_{\vec{v}}$

Rotations $R_{\theta, \vec{u}}$.

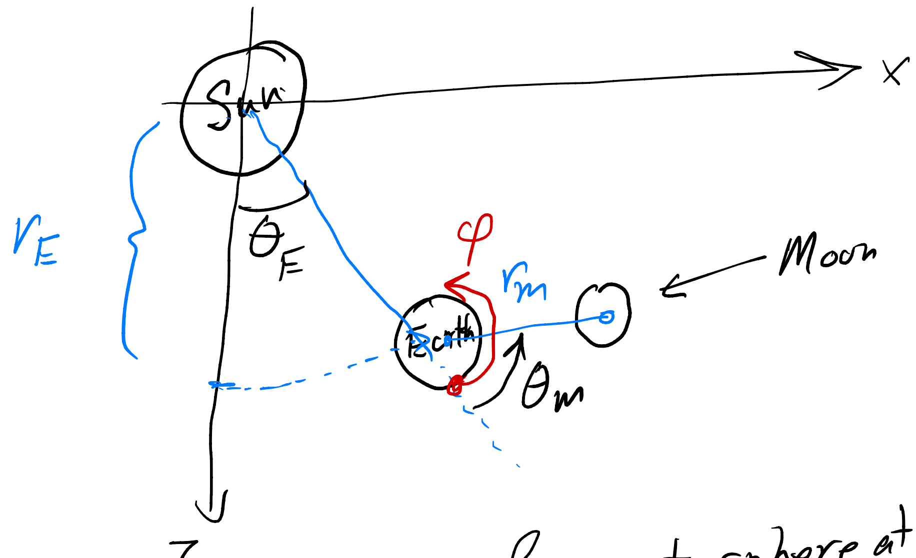
Scaling $S_{(\alpha, \beta, \gamma)}$ with $\alpha, \beta, \gamma > 0$

(not reflections!)

are all orientation preserving

Combining transformations

Suppose we are a modeling a solar system.
Sun, Earth, Moon. Use a "top view" (looking down the y-axis)



$\mathcal{S}$ - unit sphere at $\vec{O}$
(will to render Sun Earth & Moon)

θ_E - angle Earth is revolved around the Sun

θ_m - angle Moon has revolved around Earth

r_E, r_m - radii of the orbits

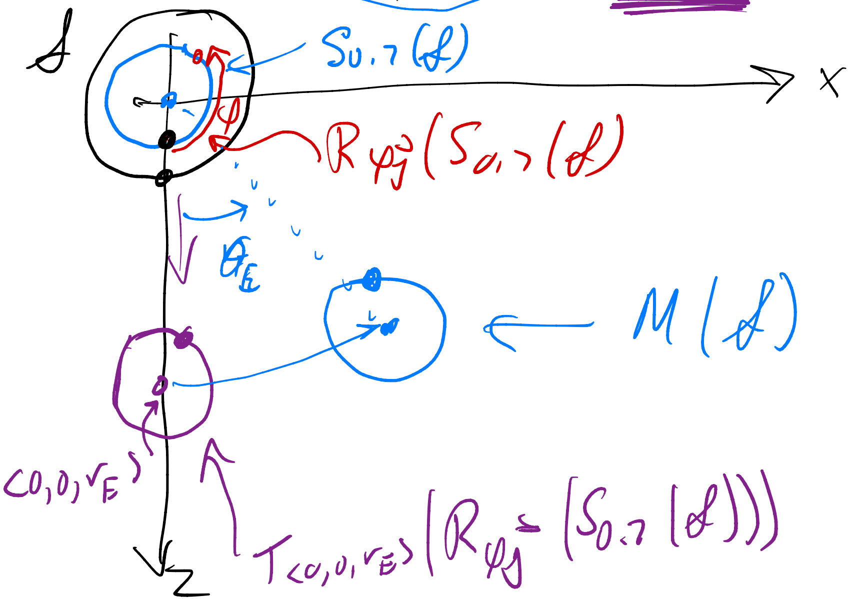
φ - how much Earth is rotated on its axis.

① Render Sun as just $\mathcal{S}$.

② Render Earth as a scaled, rotated, translated version of $\mathcal{S}$.

Let $M := R_{\theta_E, \vec{j}} \circ T_{\langle 0,0,r_E \rangle} \circ R_{\varphi, \vec{j}} \circ S_{0.7}$.

Render Earth as $M(\mathcal{S})$

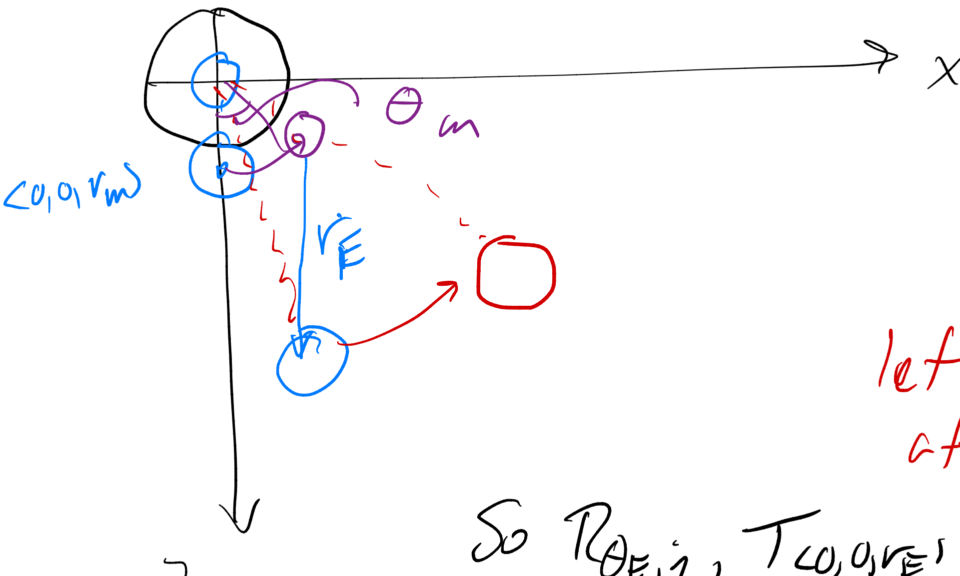


$= R_{\theta_E,\vec{j}}(T_{\langle 0,0,r_E \rangle}(R_{\varphi,\vec{j}}(S_{0.7}(\mathcal{S}))))$

For moon Use:

$$M_{moon} := R_{\theta_E, \vec{j}} \circ T_{\langle 0,0,r_E \rangle} \circ R_{\theta_m, \vec{j}} \circ T_{\langle 0,0,r_m \rangle} \circ S_{0.4}$$

Render moon as $M_m(d)$



Last two operation

$$R_{\theta_E, \vec{j}} \circ T_{\langle 0,0,r_E \rangle}$$

let the Earth's transformation affect the moon also.

So $R_{\theta_E, \vec{j}}$, $T_{\langle 0,0,r_E \rangle}$ affect the whole Earth system