Last time $\mathcal{S}$ - unit sphere

Earth transformed by

$$\underbrace{R_{\theta_{E},\vec{j}} \circ T_{\langle 0,0,r_E\rangle}}_{\text{EarthPosMatrix controls the Earth-Moon system}} \circ \underbrace{R_{\varphi,\vec{j}} \circ S_{0.7}}_{\text{Earth revolving on its axis + resizing}}$$

Moon transformed by

$$\underbrace{R_{\theta_{E},\vec{j}} \circ T_{\langle 0,0,r_E\rangle}} \circ R_{\theta_{m},\vec{j}} \circ T_{\langle 0,0,r_m\rangle} \circ S_{0.24}.$$

Pseudo-code:

LinearMapR4's $M_S, M_{E_0}, M_{E_1}, M_m$

$M_S = \text{Identity}$

Render $M_S(\mathcal{A})$; //Sun

$M_{E_0} = M_S$

$M_{E_0} = M_{E_0} \cdot R_{\theta_E, \vec{j}}$

$M_{E_0} = M_{E_0} \cdot T_{\langle 0,0,r_E \rangle}$

$M_{E_1} = M_{E_0}$

$M_{E_1} = M_{E_1} \cdot R_{\varphi, \vec{j}}$

$M_{E_1} = M_{E_1} \cdot S_{0.7}$

Render $M_{E_1}(\mathcal{A})$ //Earth

$M_m = M_{E_0}$

$M_m = M_m \cdot R_{\theta_m, \vec{j}}$

$M_m = M_m \cdot T_{\langle 0,0,r_m \rangle}$

M_S.SetIdentity();

//Render the sun w/ M_S

$M_{E_0} = M_S$

M_{E_0}.Mult_glRotate(θ_E, 0, 1, 0);

M_{E_0}.Mult_glTranslate(0, 0, r_E);

$M_{E_1} = M_{E_0}$;

M_{E_1}.Mult_glRotate(φ, 0, 1, 0);

M_{E_1}.Mult_glScale(0.7);

// Render the earth w/ M_{E_1}

$M_m = M_m \cdot S_{0.4}$

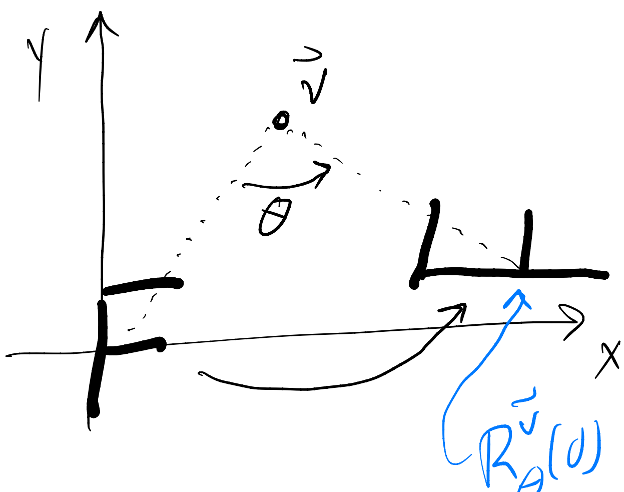
Render the $M_m(\mathcal{A})$ // Moon

Back to rigid, orientation-preserving transformations

In $\mathbb{R}^2$. Recall R_θ - rotation angle θ, CCW around $\vec{O}$

R_θ - linear. rigid, orientation preserving

Generalized rotation in $\mathbb{R}^2$:



$R_\theta^{\vec{v}}$ $\quad \vec{v} \in \mathbb{R}^2$

$R_\theta^{\vec{v}}(\vec{v}) = \vec{v}$. - Affine.

Express $R_\theta^{\vec{v}}$ as a composition of R_θ's, $T_{\vec{u}}$, S_α's.

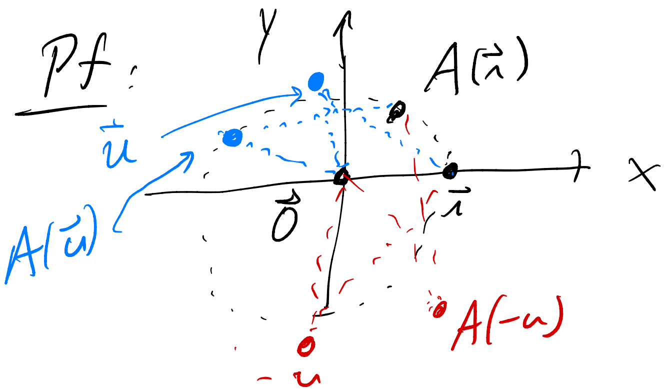
$$\boxed{R_\theta^{\vec{v}} = T_{\vec{v}} \circ R_\theta \circ T_{-\vec{v}}}$$

What are the possible rigid, orientation preserving maps in $\mathbb{R}^2$?

<u>Theorem</u> If $A: \mathbb{R}^2 \to \mathbb{R}^2$ is ROP, and if $A(\vec{0}) = \vec{0}$, then A is equal to R_θ for some $\theta \in \mathbb{R}$.

← rigid, orientation preserving

<u>Pf:</u>



$A(\vec{0}) = \vec{0}$

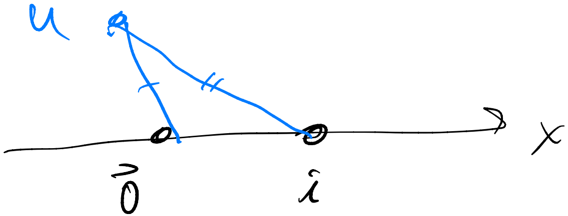
$\|A(\vec{i})\| = 1$ by rigidity

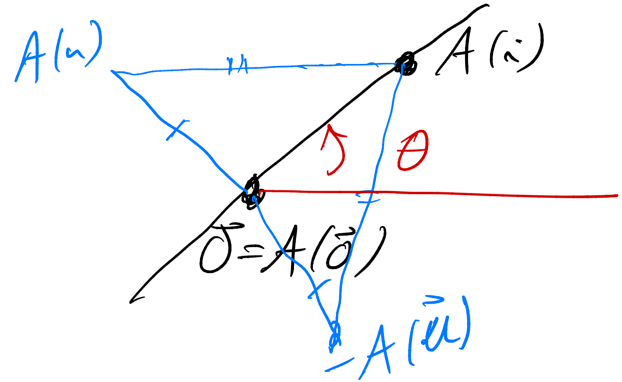
$\|\vec{i} - \vec{0}\| = \|A(\vec{i}) - A(\vec{0})\|$

$\vec{u}$ – an arbitrary point in plane.

By rigidity, $\|A\vec{u}\|$ and $\|A(\vec{u}) - A(\vec{i})\|$ are equal $\|u\|$ and of $\|u - \vec{i}\|$.

Only possibility for $A(\vec{u})$ is $\vec{u}$ rotated by the same angle as $A(\vec{i})$ was rotated. Rigidity would allow $-A(\vec{u})$ as the value, but that is prohibited by orientation preserving





Q.E.D.

Same argument shows:

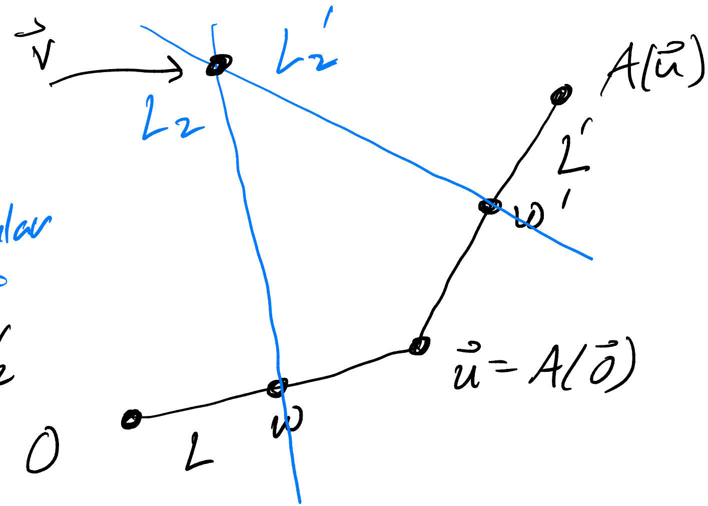
Thm If A is an ROP, $A: \mathbb{R}^2 \to \mathbb{R}^2$, and if $A(\vec{v}) = \vec{v}$, then $A = R_\theta^{\vec{v}}$ for some θ.

Theorem If $A:\mathbb{R}^2\to\mathbb{R}^2$ is ROP, then either:

(a) A is a translation $T_{\vec{u}}$, or

(b) A is a generalized rotation $R_\theta^{\vec{v}}$.

Pf: Let $\vec{u}=A(\vec{0})$. Consider $A(\vec{u})$. If $\vec{u}=0$, $A(0)=0$ so A is a rotation.



L_2, L_2' are perpendicular bisectors

$A: L_2 \mapsto L_2'$

So $A(\vec{v})=\vec{v}$

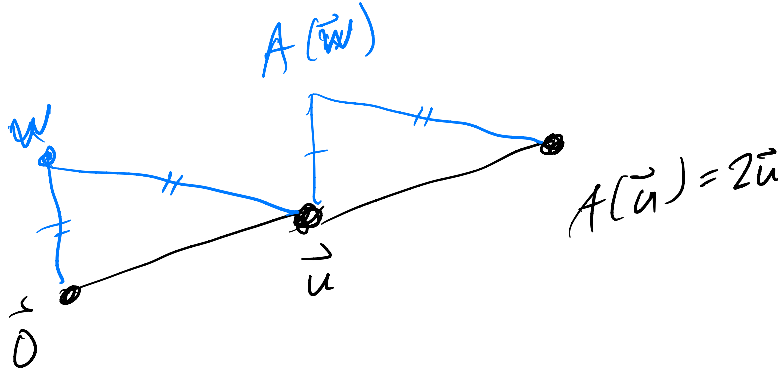
w, w' - midpoints of L, L'

Case (1) $A(\vec{u})=2\vec{u}$, then A is $T_{\vec{u}}$

Case (2) $A(\vec{u})=0$.

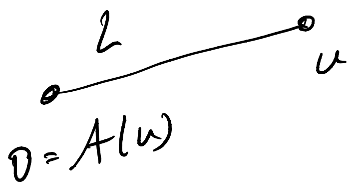
Case (3) Otherwise... we're in the situation of this picture

Case 1



In this case A is $T_{\vec u}$

Case 2



So $A(\frac{1}{2}\vec u) = \frac{1}{2}\vec u$. So, for $\vec v = \frac{1}{2}\vec u$, $A(\vec v) = \vec v$

So A is R_θ^{v}.

Case 3 $\vec v$ - intersection of L_2, L_2'

and $A(\vec v) = \vec v$.

So A is $R_\theta^{\vec v}$ for some θ.

In $\mathbb{R}^3$

Theorem: If $A: \mathbb{R}^3 \to \mathbb{R}^3$ and $A(\vec{0}) = \vec{0}$ and A is ROP, then A is a rotation $R_{\theta, \vec{u}}$ for some axis $\vec{u}$ (the axis is through the origin) and some $\theta \in \mathbb{R}$.

Example: Suppose $A(\vec{0}) = \vec{0}$ $\quad A(\vec{i}) = \vec{j}$, $A(\vec{j}) = \vec{k}$, $A(\vec{k}) = \vec{i}$

I.e. $A(\langle x, y, z \rangle) = \langle z, x, y \rangle$.

Express A in the form $R_{\theta, \vec{u}}$.

$\vec{u} = \langle 1, 1, 1 \rangle$, $\quad \theta = 120^\circ$

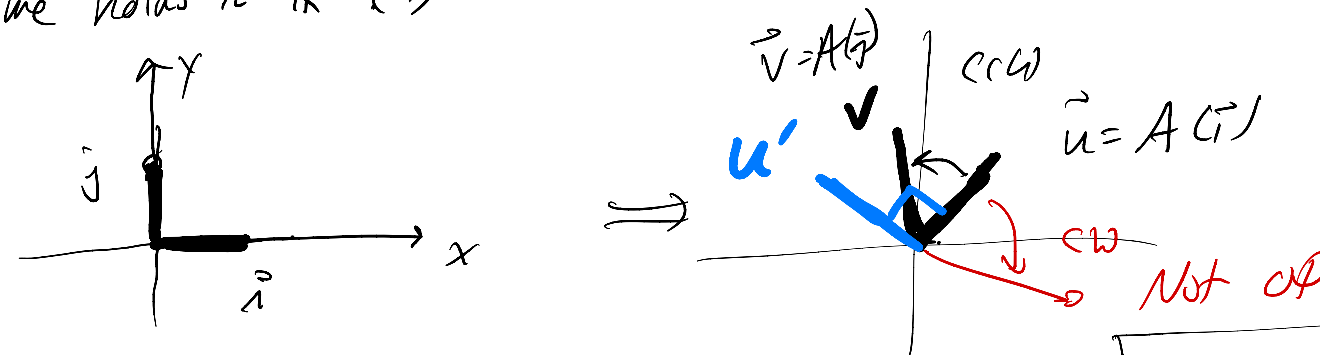
$\vec{u} = \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

same axis, but a unit.

Theorem Let $A: \mathbb{R}^2 \to \mathbb{R}^2$ be linear, with 2×2 matrix M

Then A is orientation preserving iff $\det(M) > 0$

Same holds in $\mathbb{R}^3$ (!)



u' is $\vec{u}$ rotated by 90^0.

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $\quad \vec{u}' = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$

We have: $\vec{v}$ is "to the left of $\vec{u}$" iff $\vec{u}' \cdot \vec{v} > 0$

$$\boxed{\begin{array}{l} \mathbf{u' \cdot v} \text{ is} \\ -u_2 \cdot v_1 + u_1 \cdot v_2 \\ = \\ \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \end{array}}$$