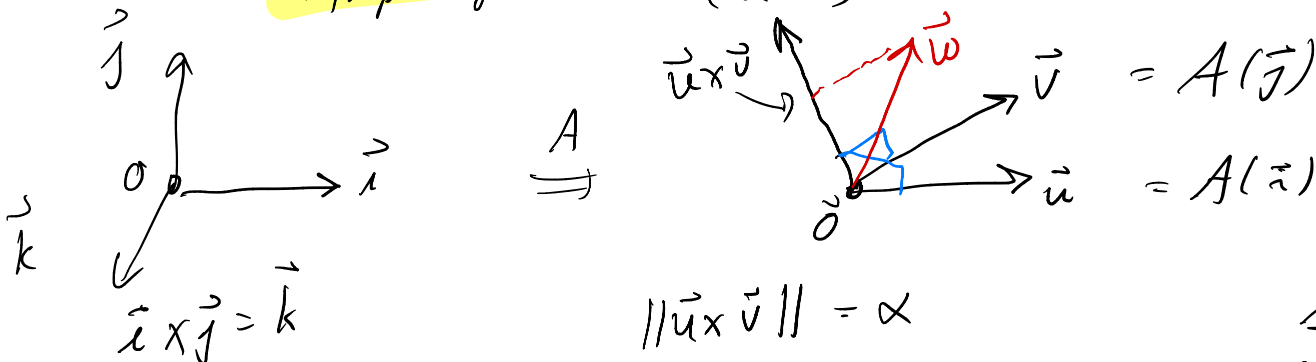
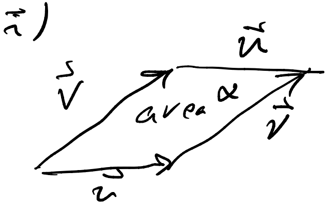
Rigid, orientation preserving (ROP) linear maps in $\mathbb{R}^3$

Linear map $\vec{x} \longmapsto M\vec{x}$ $\quad M$ - 3×3 matrix

$M = (\vec{u}, \vec{v}, \vec{w})$ $\quad \vec{u} = M\vec{i} \quad \vec{v} = M\vec{j} \quad \vec{w} = M\vec{k}$

Defn: M is orientation preserving if the triple product $(\vec{u}\times\vec{v})\cdot\vec{w} > 0$.





$\|\vec{u}\times\vec{v}\| = \alpha$

$(\vec{u}\times\vec{v})\cdot\vec{w} > 0$ means projection of $\vec{w}$ onto $\vec{u}\times\vec{v}$ is in the positive direction of $\vec{u}\times\vec{v}$.

$\underline{\text{Fact}}$ $\det(M) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - u_3 v_2, \; u_3 v_1 - u_1 v_3, \; u_1 v_2 - u_2 v_3 \rangle$$

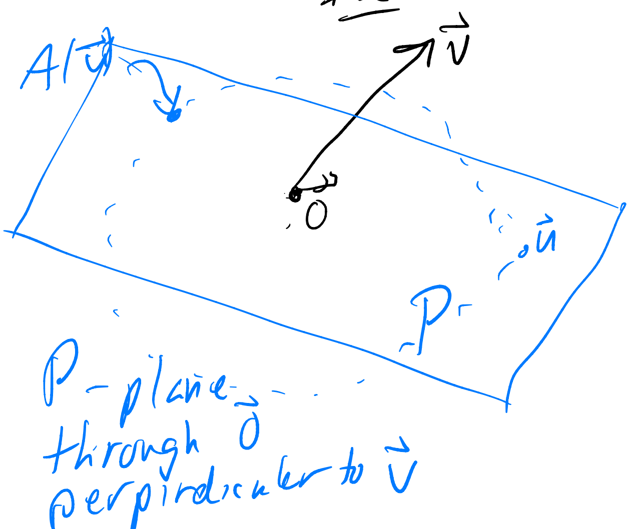
$(\vec{u} \times \vec{v}) \cdot \vec{w}$ yields the same result as $\det(M)$

det = "determinant"

Euler's Rotation Theorem - If A is a ROP transformation of $\mathbb{R}^3$ and $A(\vec{0}) = \vec{0}$, then $A = R_{\theta,\vec{u}}$ for some $\theta \in \mathbb{R}$, $\vec{u} \in \mathbb{R}^3$.

Proof. Suffices to show that $A(\vec{v}) = \vec{v}$ for some $\vec{v} \neq 0$.

Idea $\vec{v}$ is the rotation axis.



By rigidity of A, A maps P to P.

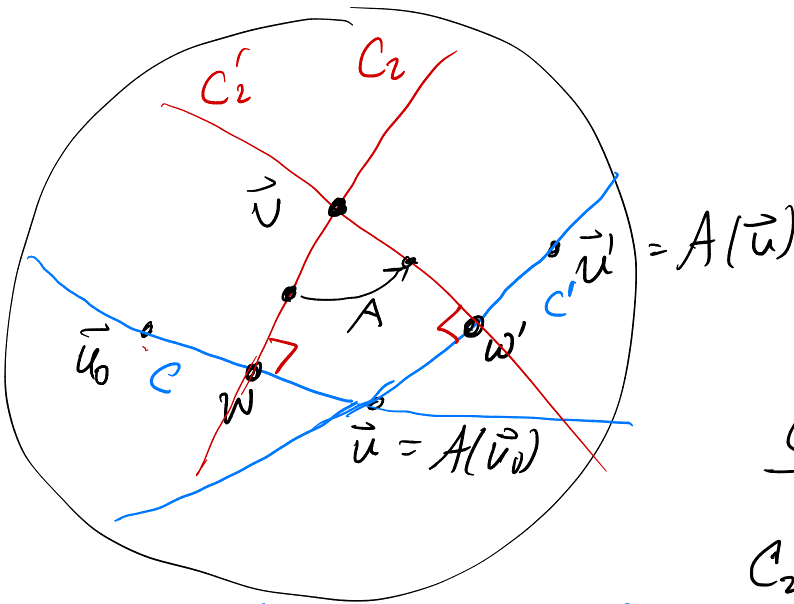
So A restricted to the plane is a rotation around $\vec{0}$.

So A is a rotation around $\vec{v}$ (in $\mathbb{R}^3$)

Let's prove there is a $\vec{v} \neq \vec{0}$ s.t. $A(\vec{v}) = \vec{v}$.

There must be some $\vec{u}_0$ s.t. $A(\vec{u}_0) \neq -\vec{u}_0$. (wlog, $\|\vec{u}_0\| = 1$)

Because, A is orientation-preserving



C = great circle joining $\vec{u}_0$ and $\vec{u}$

C' = " " " $\vec{u}$ and $\vec{u}'$

Case 1 $\vec{u}' = \vec{u}_0$

By rigidity, $A(\vec{w}) = \vec{w}$

$\vec{w}$ is the midpoint of $\vec{u}_0$ & $\vec{u}$.

Take $\vec{v} = \vec{w}$.

Case (2) $\vec{u}' \neq \vec{u}_0$

w' - midpoint of $\vec{u}$ and u'

C_2, C_2' - perpendicular great circles

$A: C_2 \mapsto C_2'$ since A is ROP.

$\vec{v}$ - intersection of C_2 & C_2' $\boxed{A(\vec{v}) = \vec{v}}$

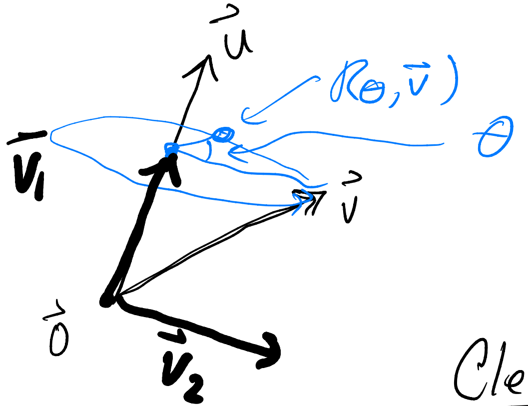
The general formula for the matrix representation of the rotation $R_{\theta,\mathbf{u}}$ is quite complicated: $R_{\theta,\vec{u}}$ $c = \cos\theta$ $s = \sin\theta$ $\vec{u} = \langle u_1, u_2, u_3 \rangle$ & is a unit vector.

$$\begin{pmatrix} (1-c)u_1^2 + c & (1-c)u_1u_2 - su_3 & (1-c)u_1u_3 + su_2 & 0 \\ (1-c)u_1u_2 + su_3 & (1-c)u_2^2 + c & (1-c)u_2u_3 - su_1 & 0 \\ (1-c)u_1u_3 - su_2 & (1-c)u_2u_3 + su_1 & (1-c)u_3^2 + c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \text{(II.10)}$$

The formula (II.10) for $R_{\theta,\mathbf{u}}$ will be derived later in Section II.3.6. This

The matrix formula for $R_{\theta,\vec{u}}$

How to compute $R_{\theta,\vec{u}}(\vec{v})$?

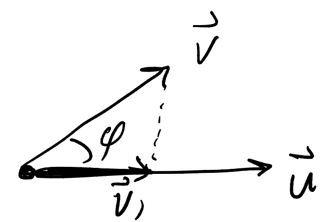


$\vec{V}_1$ – component of $\vec{v}$ parallel to $\vec{u}$

$\vec{V}_2$ - component of $\vec{v}$ perpindicular to $\vec{u}$

Clearly $R(\vec{V}_1) = \vec{V}_1$ and $\vec{V}_2 = \vec{V} - \vec{V}_1$ since $\vec{V}_1 + \vec{V}_2 = \vec{V}$.

Step(1) Find a formula for $\vec{v}_1$.



$\|u\| = 1$

Claim $\vec{V}_1 = (\vec{u}\cdot\vec{v})\vec{u}$

$\|v_1\| = \|\vec{v}\|\cdot\cos\varphi$

$\vec{u}\cdot\vec{v} = \|u\|\cdot\|v\|\cos\theta = \|v\|\cos\theta$ since $\|u\|=1$

But we need a matrix giving $\vec{v}_1$.

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\vec{u}^T = (u_1\ u_2\ u_3)$$

$$(u_1\ u_2\ u_3)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{v}_1 = (\vec{u}\circ\vec{v})\,\vec{u} = (\vec{u}^T\vec{v})\,\vec{u}$$
$$= \vec{u}\,(\vec{u}^T\vec{v})$$
$$= (\vec{u}\vec{u}^T)\,\vec{v} \quad \text{- associativity}$$

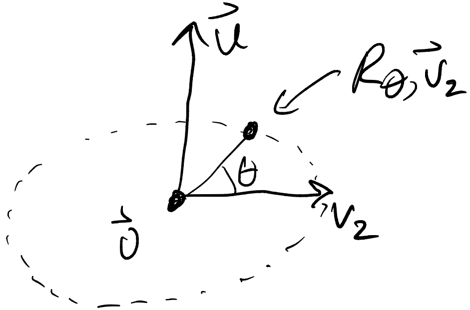
$$(uu^T) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}(u_1\ u_2\ u_3)$$

$$= \begin{pmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{pmatrix} = \mathrm{Proj}_{\vec{u}}$$

$$\vec{v}_1 = (\mathrm{Proj}_{\vec{u}})\,\vec{v}$$

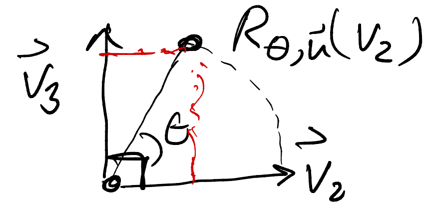
$$\vec{v}_2 = \vec{v} - \mathrm{Proj}_{\vec{u}}\,\vec{v}$$
$$= (I - \mathrm{Proj}_{\vec{u}})\,\vec{v}$$

Step 2 Rotate $\vec{v}_2$ around $\vec{u}$ (do it with a matrix!)



Top view

$\vec{u}$ - pointing out of the picture



Let $\vec{v}_3 = \vec{u} \times \vec{v}_2 = \vec{u} \times \vec{v}$ since $\vec{v} = \vec{v}_2 + \vec{v}_1$ and $\vec{u} \times \vec{v}_1 = 0$.

$\|v_3\| = \|\vec{v}_2\|$ since $\|u\| = 1$ and $\vec{v}_2$ is perpindicular to $\vec{u}$

Finally $R_{\theta,\vec{u}}(\vec{v}_2) = (\cos\theta)\,\vec{v}_2 + (\sin\theta)\,\vec{v}_3 = c\vec{v}_2 + s\vec{v}_3$

$\vec{v}_2 = \langle v_{2x}\ v_{2y}\ v_{2z} \rangle$

$\vec{v}_3 = \vec{u} \times \vec{v}_2 = \langle u_2 v_{2z} - u_3 v_{2y},\ u_3 v_{2x} - u_1 v_{2z},\ u_1 v_{2y} - u_2 v_{2x} \rangle$

$$\vec{v}_3 = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix} = (M_{u\times})\, \vec{v}_2$$

$$\vec{v}_3 = (M_{u\times})\, \vec{v}$$

<u>Step 3</u> Put it together.

$$R_{\theta,\vec{u}}(v) = \vec{v}_1 + c\,\vec{v}_2 + s\,\vec{v}_3$$

$$= (\mathrm{Proj}_{\vec{u}})\vec{v} + c\,(I - \mathrm{Proj}_{\vec{u}})\vec{v} + s\,(M_{\vec{u}\times})\,\vec{v}$$

$$= \left[\mathrm{Proj}_{\vec{u}} + c\,(I - \mathrm{Proj}_{\vec{u}})\vec{v} + s\,M_{\vec{u}\times}\right] \vec{v}$$

$$= \left[(1-c)\,\mathrm{Proj}_{\vec{u}} + cI + s\,M_{\vec{u}\times}\right] \vec{v} \quad \leftarrow \text{Tada!}$$