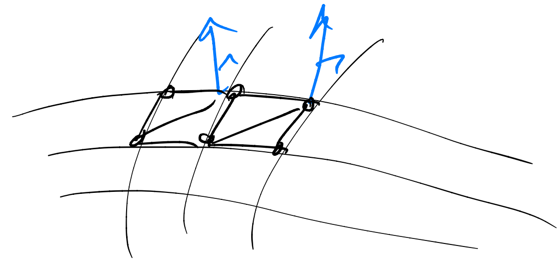
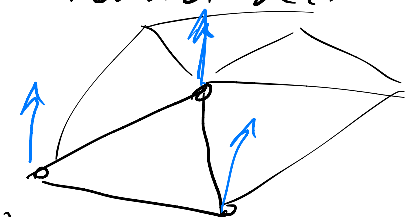
# Surfaces and Normals



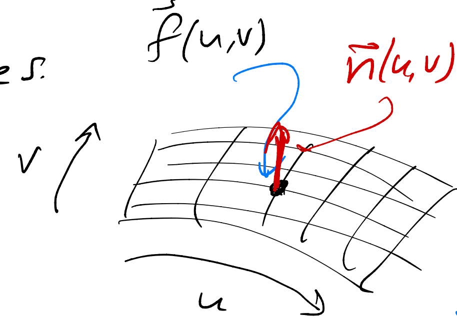
Last time - Two methods for computing normals.

Method #1: Given coordinates of vertices & a triangulation: Compute normals of triangles, average normals of triangles to estimate normal vectors at vertices.



When rendering - we will average (like smooth shading) normals from vertices to the get normals at pixels in the interior of the triangle. (Phong interpolation or Phong shading)

# Method #2: Parametric Surfaces:

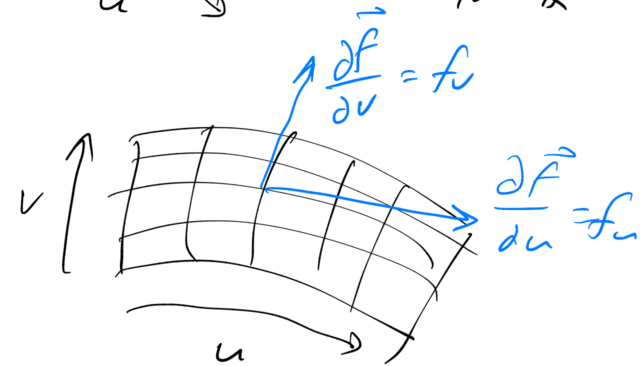



$u, v \in \mathbb{R}$

$\vec{f}(u,v)$ is a vector in $\mathbb{R}^3$

$\vec{f}(u,v)$ - defines a surface

$\vec{n}(u,v)$ - normal at the the point $\vec{f}(u,v)$.

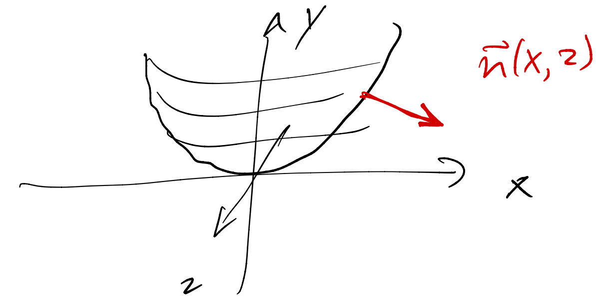



Let $\vec{n}(u,v) = \frac{\partial \vec{f}}{\partial u} \times \frac{\partial \vec{f}}{\partial v} = \vec{f}_u \times \vec{f}_v$

or $\vec{n}(u,v) = \frac{\vec{f}_u \times \vec{f}_v}{\| \vec{f}_u \times f_v \|}$.

## Example Paraboloid

$$y = x^2 + z^2.$$




x and z are the parameters
(i.e. instead of being called u, v)

$$\vec{f}(x,z) = \langle x,\ x^2+z^2,\ z \rangle.$$

$$\vec{f}_x = \frac{\partial \vec{f}}{\partial x} = \langle 1,\ 2x,\ 0 \rangle \qquad \vec{f}_z = \frac{\partial \vec{f}}{\partial z} = \langle 0,\ 2z,\ 1 \rangle$$

$$\vec{f}_x \times \vec{f}_z = \underline{\langle 2x,\ -1,\ 2z \rangle}$$

Answer is in terms of x, z
(i.e. in terms of u, v).

Be careful about the sign.

Alternative parameterization

$r = \sqrt{x^2+z^2}$, $\theta$ - angle around y-axis

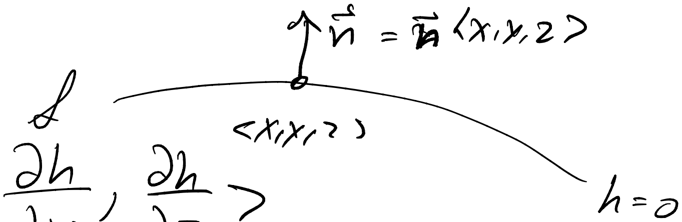
$\theta = \operatorname{atan}(z/x)$

$$f(r,\theta) = \langle r\sin\theta,\ r^2,\ r\cos\theta \rangle$$

## Method #3: Level Set definition of a surface.

Let $h(x,y,z)$ be a scalar-valued function

The surface $\mathcal{S} = \{ \langle x,y,z \rangle : h(x,y,z) = 0 \}$




Let $\vec{n}(x,y,z) = \nabla h = \left\langle \dfrac{\partial h}{\partial x}, \dfrac{\partial h}{\partial y}, \dfrac{\partial h}{\partial z} \right\rangle$

– the gradient of $h$.

Example Paraboloid: $y = x^2 + z^2$. Can be expressed

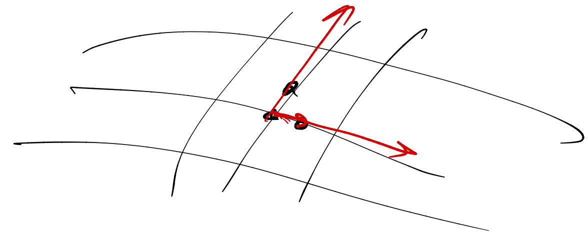
$\{ \langle x,y,z \rangle : y - x^2 - z^2 = 0 \}$ $\quad h(x,y,z) = y - x^2 - z^2$.

$\vec{n}(x,y,z) = \langle -2x, 1, -2z \rangle$. ← $\vec{n}$ is a function of $x,y,z$ in general.

Watch out for sign.

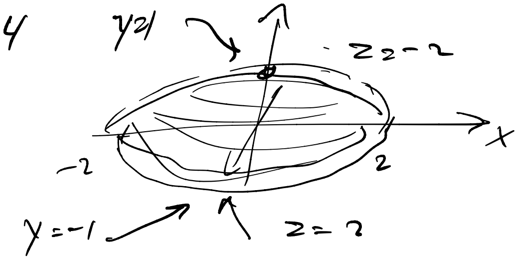
In any method it may need to reversed.

Partial derivatives -
can be calculated
when there is a
mathematical function



- otherwise use a numerical approximation.

Example Ellipsoid $x^2 + 4y^2 + z^2 = 4$




Squashed ~~the~~ sphere

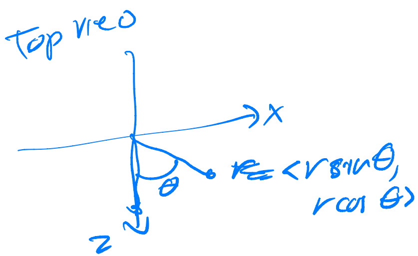
$$\left\{ \langle x, y, z\rangle : \ x^2 + 4y^2 + z^2 - 4 = 0 \right\}$$

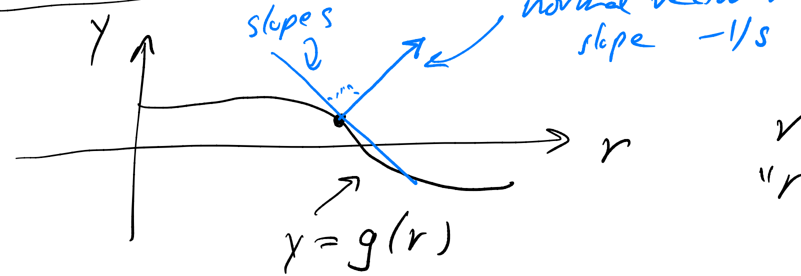
$\nabla h = \underbrace{\langle 2x, 8y, 2z \rangle}$ – will normal to the ellipsoid. at $\langle x, y, z\rangle$

$\vec{n}(x,y,z)$ as a function of $x,y,z$

given $x,y,z$ is on the ellipsoid.

## Method 2': Surface of rotation




$r = \sqrt{x^2+z^2}$
"radius"

Rotate $g$ around the $y$-axis, $g$ generates a surface of rotation

Parametric form: $f(r,\theta) = \langle r\cdot\sin\theta, g(r), r\cdot\cos\theta\rangle$

Level surface form: $h(x,y,z) = y - g(\sqrt{x^2+z^2})$

Shortcut method Slope is $s = g'(r)$

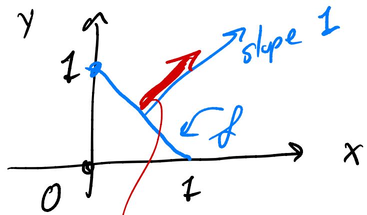
A vector perpendicular (normal) is $\langle s, -1\rangle$ (In $r$-$y$ plane)

or – better perhaps $\langle -s, 1\rangle$

When rotate around $y$-axis – normals are $\langle -s\cdot\sin\theta, 1, -s\cos\theta\rangle$

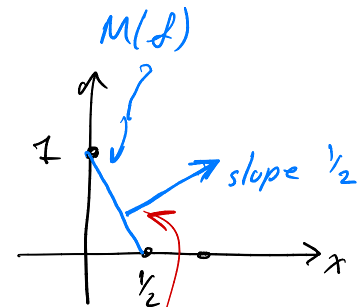
# Transformations of normals:

## Example 1: $\mathbb{R}^2$




$$\xrightarrow{S_{\langle 1/2,\, 1\rangle}}$$

$$M = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$$




$\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$ is a unit ~~vec~~ normal

or $\langle 1,1 \rangle$ is normal

$\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle$ is a unit normal

or $\langle 2,1 \rangle$ is normal

$M\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ does not yield $\langle 2, 1 \rangle$

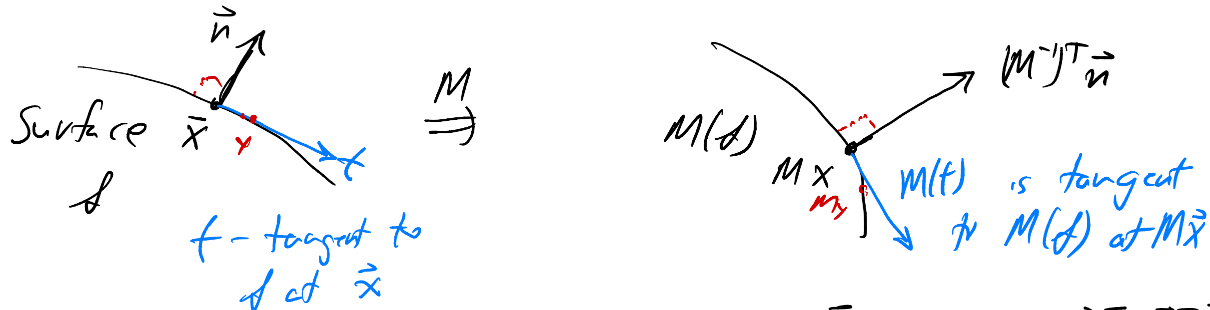
$M$ does not correctly transform normals.

Instead: $(M^{-1})^T = (M^T)^{-1}$ is used to transform normals.

**Theorem** If $\vec{n}$ is normal to surface $S$ at point $\vec{x}$ and $M$ is a $3\times 3$ matrix for a linear transformation then $(M^{-1})^T \cdot \vec{n}$ is normal to the surface $M(S)$ at $M\vec{x}$.

(Same holds for $M$ the linear part of affine transformation)

**Pf:**




So need to show: $(M^{-1})^T\vec{n} \bullet M\vec{t} = 0$ if $\vec{n} \circ \vec{t} = 0$

i.e. $((M^{-1})^T\vec{n})^T (M\vec{t}) = 0$ if $\vec{n}^T \vec{t} = 0$

$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$

$(AB)^T = B^T A^T$

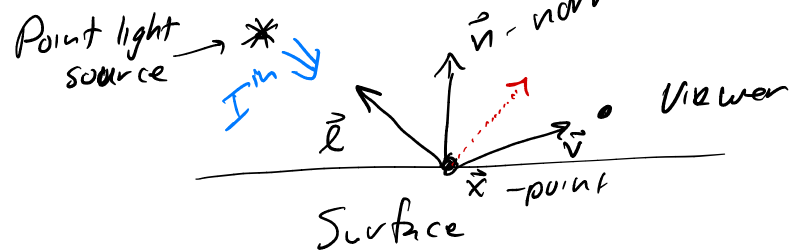
$(A^T)^T = A$

$$((M^{-1})^T\vec{n})^T (M\vec{t}) = \vec{n}^T((M^{-1})^T)^T \cdot M\vec{t} = \vec{n}^T M^{-1} M\vec{t} = \vec{n}^T\vec{t} = 0$$

by assumption

QED

# Phong lighting:




r - direction of perfect reflection

$\vec{l}, \vec{v}, \vec{n}$ - unit vectors - light direction
normal vector
view direction

Ambient light - coming from all directions
reflects in all direction

Diffuse light - coming from a point light source
reflecting equally in all direction

Specular light - coming from a point light, reflecting more-or-less mirror like.