Last time: linear interpolation $\quad (1-\alpha)\vec{x} + \alpha\vec{y} = \text{lerp}(\vec{x},\vec{y},\alpha)$

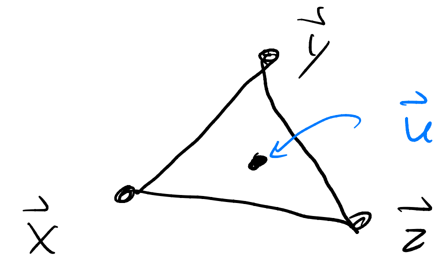
Barycentric coordinates:

3 vertices $\vec{x}, \vec{y}, \vec{z}$

non-colinear

form a triangle T

lie in a plane P

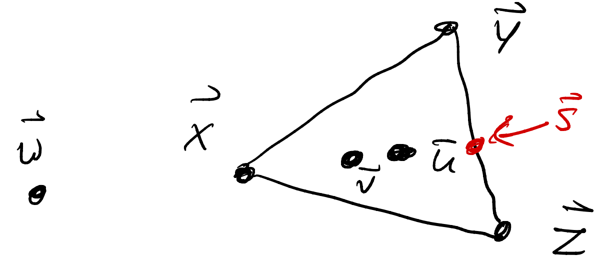


Express $\vec{u}$ in T as $\vec{u} = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}$

where $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$

or $\vec{u}$ in P, the same way but now possibly α, β, γ are negative

Example:



$$\vec{u} = \frac{1}{3}\vec{x} + \frac{1}{3}\vec{y} + \frac{1}{3}\vec{z}$$

$$\vec{v} = \frac{1}{2}\vec{x} + \frac{1}{4}\vec{y} + \frac{1}{4}\vec{z}$$

$$\vec{w} = 2\vec{x} - \frac{1}{2}\vec{y} - \frac{1}{2}z$$

$$\vec{s} = \frac{1}{2}\vec{y} + \frac{1}{2}\vec{z} = \mathrm{lerp}\left(\vec{y}, \vec{z}, \tfrac{1}{2}\right)$$

$$\vec{v} = \mathrm{lerp}\left(\vec{s}, \vec{x}, \tfrac{1}{2}\right) = \frac{1}{2}\vec{x} + \frac{1}{2}\vec{s} = \frac{1}{2}\vec{x} + \frac{1}{4}\vec{y} + \frac{1}{4}\vec{z}$$

Definition Let $\vec{x}_1 \cdots \vec{x}_k \in \mathbb{R}^d$.

A linear combination of $\vec{x}_1 \cdots \vec{x}_k$ is of the form $\alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_k\vec{x}_k$

It is an affine combination if $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$

It is a weighted average if also $\alpha_1 \geq 0, \alpha_2 \geq 0 \cdots, \alpha_k \geq 0$

So $\vec{u}$ & $\vec{v}$ are weighted averages of $\vec{x}, \vec{y}, \vec{z}$.

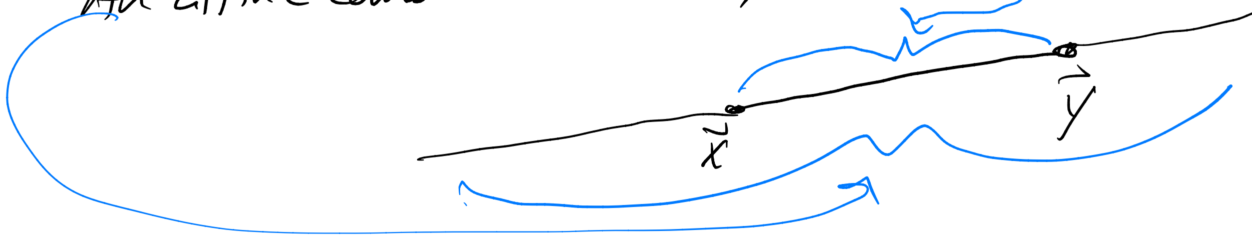
w is an affine combination of $\vec{x}, \vec{y}, \vec{z}$

Lerping

$$\text{lerp}(\vec{x},\vec{y},\alpha) = (1-\alpha)\vec{x} + \alpha\vec{y}$$

Weighted average if $\alpha \in [0,1]$

An affine combination in any case



Theorem Let $\vec{x},\vec{y},\vec{z}$, T, P be as above

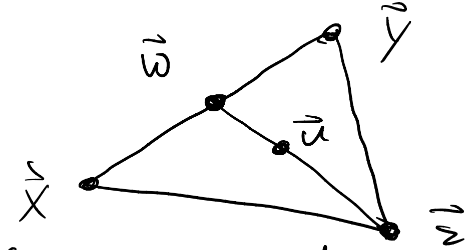
ⓐ For any $\vec{u}$ in or on T, $\vec{u}$ is a weighted average

$\vec{u} = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}$ of $\vec{x},\vec{y},\vec{z}$

α,β,γ - barycentric coordinates of $\vec{u}$

ⓑ For any $\vec{u}$ in P, $\vec{u}$ can be written uniquely as an affine combination of $\vec{x},\vec{y},\vec{z}$ as $\vec{u} = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}$.

Proof of (a) If $\vec{u} = \vec{z}$, $\alpha = \beta = 0$, $\gamma = 1$



Otherwise the line containing $\vec{u}$ and $\vec{z}$ intersects $\overline{\vec{x}\vec{y}}$ at $\vec{w}$.

$\vec{w} = a\vec{x} + b\vec{y}$ for some $a = 1-b$, $a, b \geq 0$ $\quad a+b=1$

since $\vec{w} = \text{lerp}(\vec{x}, \vec{y}, b)$

Now $\vec{u} = c\vec{w} + d\vec{z}$ for some c, d, $\quad c+d=1$, $c, d \geq 0$

So $\vec{u} = (ac)\vec{x} + (bc)\vec{y} + d\vec{z}$

$\alpha = ac$, $\beta = bc$, $\gamma = d$ gives $\vec{u} = \alpha\vec{x} + \beta\vec{y} + \gamma\vec{z}$.

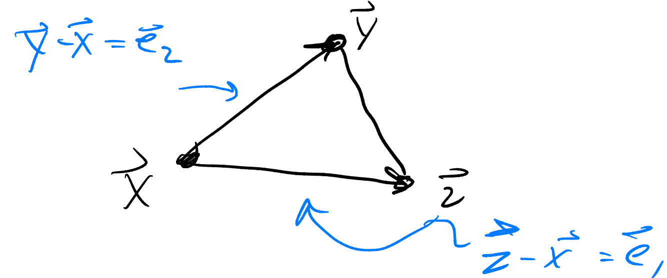
Since $a+b=1$, $c+d=1$, get $ac + bc + d = c + d = 1$

and $\alpha, \beta, \gamma \geq 0$ since $a, b, c, d \geq 0$.

Qed (a)

$\text{lerp}(\vec{x}, \vec{y}, b) = \underbrace{(1-b)}_{a}\vec{x} + \underbrace{b}_{b}\vec{y}$, equals y if $b=1$, equals x if $b=0$.

Proof of b: Let $\vec{u}$ be in P.



$\vec{y}-\vec{x}$ and $\vec{z}-\vec{x}$ are linearly independent.

So every point $\vec{u}$ in the plane P can be uniquely expressed as

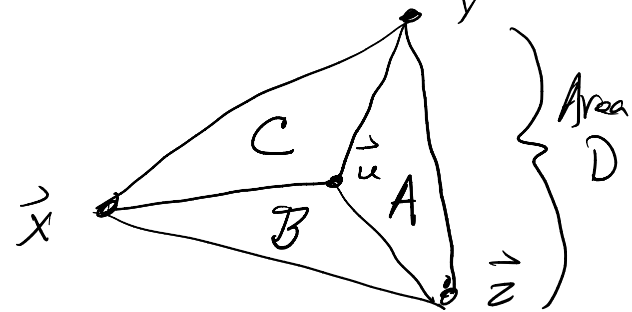
$$\vec{u} = \vec{x} + a(\vec{y}-\vec{x}) + b(\vec{z}-\vec{x})$$

$$= \underbrace{(1-a-b)}_{\alpha = 1-a-b}\vec{x} + \underbrace{a}_{\beta = a}\vec{y} + \underbrace{b}_{\gamma = b}\vec{z}$$

Q.E.D.

Question: Given $\vec{x}, \vec{y}, \vec{z}$, and $\vec{u}$ how can we find α, β, γ the barycentric coordinates of $\vec{u}$?

Method #1: Area interpretation of barycentric coordinates



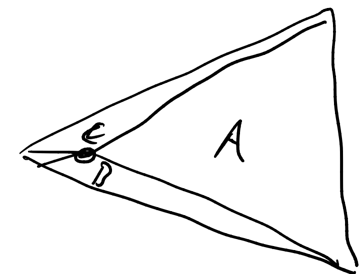
Then

$$\alpha = \frac{A}{D}$$

$$\beta = \frac{B}{D}$$

$$\gamma = \frac{C}{D}$$

Proof:

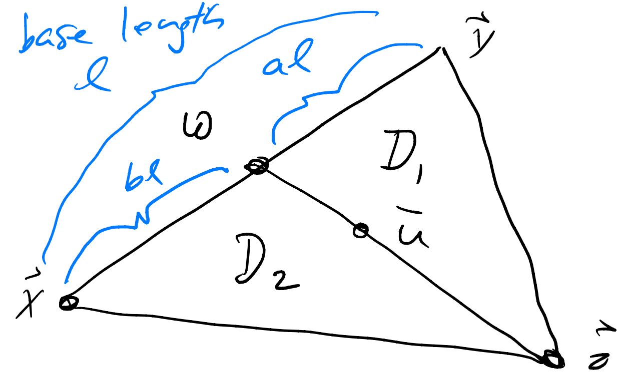


Proof: (See proof of (a) above)

$$D_1 + D_2 = D$$

$$w = a\vec{x} + b\vec{y} = \mathrm{lerp}(\vec{x}, \vec{y}, b)$$

$$\begin{cases} D_1 = aD & a = \dfrac{D_1}{D} \\ D_2 = bD & b = \dfrac{D_2}{D} \end{cases}$$



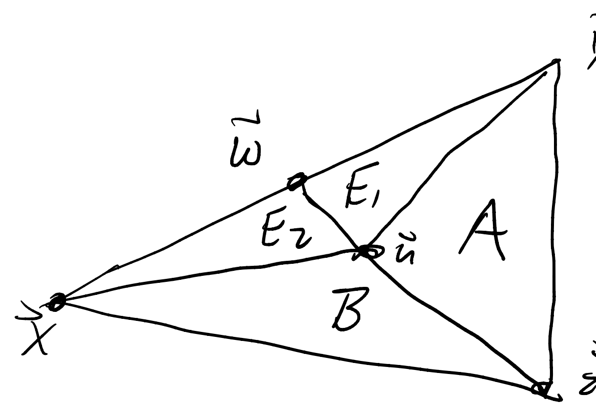
Now $\vec{u} = c\vec{w} + d\vec{z}$

so $A = cD_1$ $\quad c = \dfrac{A}{D_1}$

$B = cD_2$ $\quad c = \dfrac{B}{D_2}$

$$\left.\begin{aligned} E_1 &= dD_1 & d &= \frac{E_1}{D_1} \\ E_2 &= dD_2 & d &= \frac{E_2}{D_2} \end{aligned}\right\} d = \frac{E_1 + E_2}{D_1 + D_2}$$

so $d = \dfrac{C}{D}$



$$\vec{u} = (ac)\vec{x} + (bc)\vec{y} + d\vec{z} = \frac{A}{D_1}\cdot\frac{D_1}{D}\vec{x} + \frac{B}{D_2}\cdot\frac{D_2}{D}\vec{y} + \frac{C}{D}\vec{z} = \frac{A}{D}\vec{x} + \frac{B}{D}\vec{y} + \frac{C}{D}\vec{z}.$$

Application: Interpolation of function values

Given $f(\vec{x}) = c \quad f(\vec{y}) = d \quad f(\vec{z}) = e$

Interpolate values of f to the triangle T

or extrapolate to the plane P

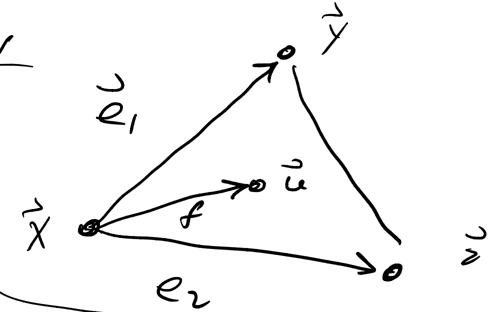
by: $\quad$ if $\vec{u} = \alpha \vec{x} + \beta \vec{y} + \gamma \vec{z}$,

estimate/interpolate

$$f(\vec{u}) = \alpha c + \beta d + \gamma e$$
$$= \alpha f(\vec{x}_1) + \beta f(\vec{x}_2) + \gamma f(\vec{x}_3).$$

"No perspective interpolation"

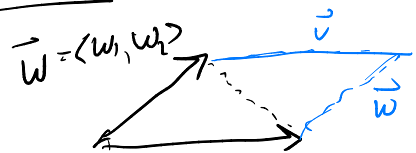
Example of computing barycentric coordinates



$\vec{e}_1 = \vec{y} - \vec{x} \qquad \vec{e}_2 = \vec{z} - \vec{x}$

$\vec{f} = \vec{u} - \vec{x}$

Basic principle (in $\mathbb{R}^2$)



$\vec{v} = \langle v_1\ v_2 \rangle$

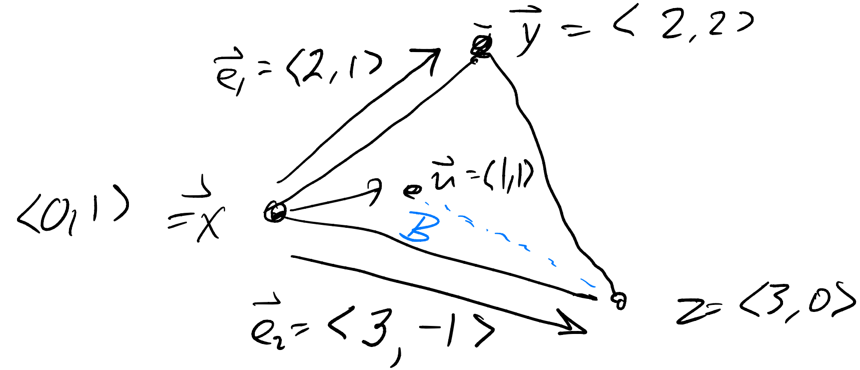
$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

is the area of the parallelogram

Twice the area of the triangle.

In $\mathbb{R}^3$, can use $\|\vec{v} \times \vec{w}\|$ - but then have to watch for sign of the area.

Example:



$$\text{Area } D = \frac{1}{2}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} = \frac{1}{2}\cdot 5$$

$$\text{Area } B = \frac{1}{2}\begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} = \frac{1}{2}\cdot 1$$

$\bar{f} = \langle 1,0 \rangle$

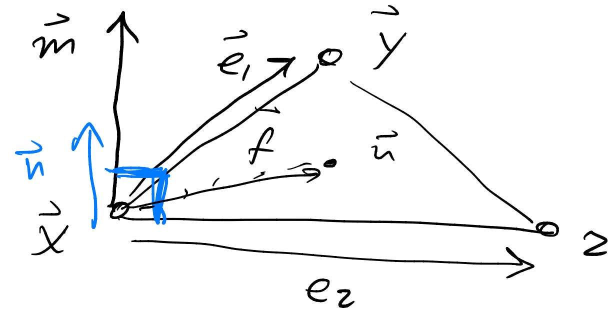
$$\text{Area } C = \frac{1}{2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}\cdot 1$$

So $\beta = \frac{B}{D} = \frac{1}{5}$ $\quad \gamma = \frac{C}{D} = \frac{1}{5}$

$$\alpha = 1 - \beta - \gamma = \frac{3}{5}.$$

Good in $\mathbb{R}^2$ for handcomputation if $\vec{n}$ does lie in the plance

Vector method (Any dimension $d \geq 2$, good for computers)
(Is robust if $\vec{u}$ is not exactly in the plane.)



$$\vec{m} = \text{component of } \vec{e}_1 \text{ perpendicular to } \vec{e}_2$$
$$= \vec{e}_1 - (\text{component of } \vec{e}_1 \text{ parallel to } \vec{e}_2)$$
$$= \vec{e}_1 - \frac{(\vec{e}_1 \circ \vec{e}_2)\vec{e}_2}{\|e_2\|^2} = \vec{e}_1 - \frac{(\vec{e}_1 \cdot \vec{e}_2)\vec{e}_2}{e_2^2}$$

$$\left(e_2^2 = \vec{e}_2 \cdot \vec{e}_2 = \|e_2\|^2\right)$$

$$\vec{m}^* = e_2^2\, \vec{m} = \underline{e_1 e_2^2 - (\vec{e}_1 \cdot \vec{e}_2)\vec{e}_2}$$

$$\vec{n} = \frac{\vec{m}}{\|\vec{m}\|} = \frac{m^*}{\|m^*\|}$$

$\vec{e}_1 \cdot \vec{n}$ - height of the triangle

$\|e_2\|$ - base length

Area $D = \frac{1}{2}\vec{e}_1 \cdot \vec{n}\, \|e_2\|$

Similarly $B = \frac{1}{2}\vec{f} \cdot \vec{n}\, \|e_2\|$

So $$\beta = \frac{B}{D} = \frac{\vec{n} \circ \vec{f}}{\vec{n} \cdot \vec{e}_1} = \frac{\vec{m}^* \cdot \vec{f}}{\vec{m}^* \cdot \vec{e}_1} = \frac{\vec{e}_1 \vec{e}_2^{\,2} - (\vec{e}_1 \cdot \vec{e}_2)\vec{e}_2}{e_1^2 e_2^2 - (\vec{e}_1 \circ \vec{e}_2)^2} \cdot \vec{f}$$

$\beta = \vec{u}_\beta \cdot \vec{f}$ where $\vec{u}_\beta = \dfrac{\vec{e}_2^{\,2}\,\vec{e}_1 - (\vec{e}_1\cdot\vec{e}_2)\,\vec{e}_2}{\vec{e}_1^{\,2}\,\vec{e}_2^{\,2} - (\vec{e}_1\,\vec{e}_2)^2}$

u_β - perpendicular to $\overline{\vec{x}\vec{z}}$

and scaled so that $\underline{\vec{u}_\beta \cdot (\vec{y}-\vec{x}) = 1}$