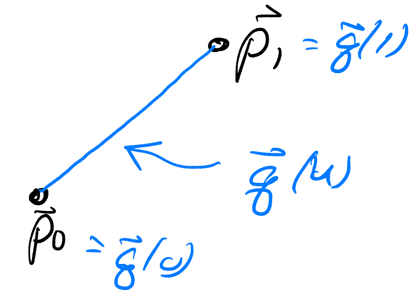
Bézier curves:

Specify a smooth curve $\vec{g}(u)$ $\quad u \in [0,1]$ $\quad \vec{g}(u) \in \mathbb{R}^k$ for $k \geq 1$.

Degree 1 Bezier curve.

Has 2 control points $\vec{p}_0, \vec{p}_1$

$$\vec{g}(u) = \operatorname{lerp}(\vec{p}_0, \vec{p}_1, u)$$

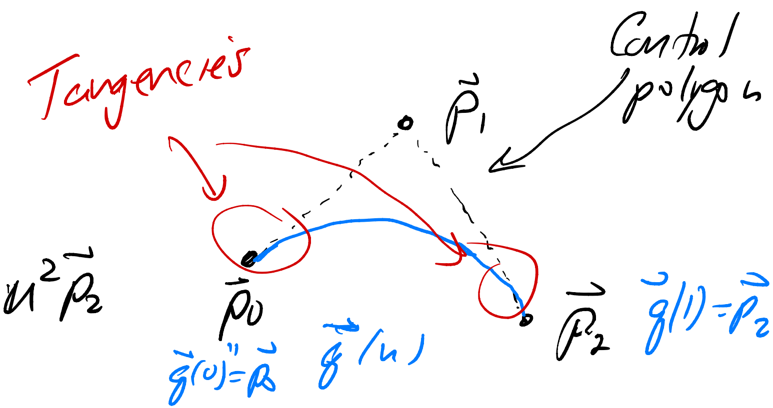


Degree 2 Bezier curve

3 control points

$\vec{p}_0, \vec{p}_1, \vec{p}_2 \in \mathbb{R}^k$

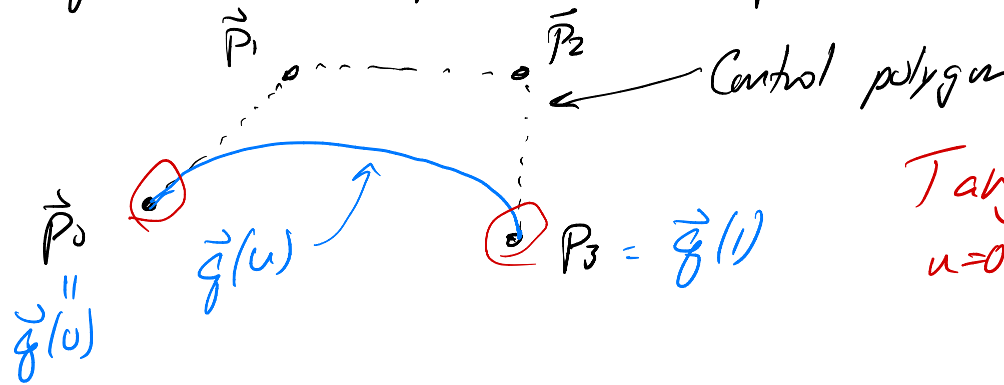
$$\vec{g}(u) = (1-u)^2\vec{p}_0 + 2u(1-u)\vec{p}_1 + u^2\vec{p}_2$$



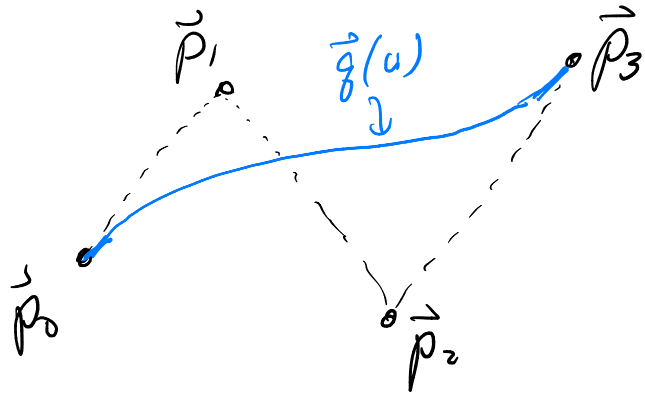
Degree 3 Bezier curves

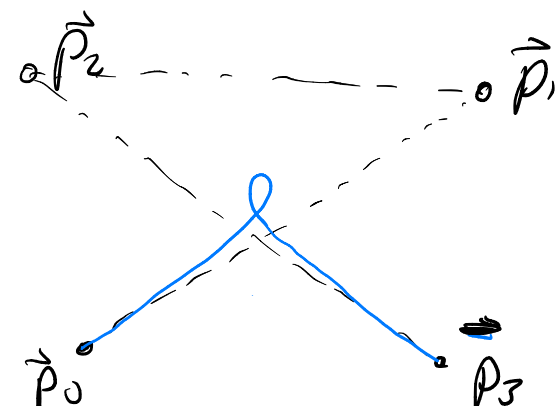
4 control points $\vec{p}_0, \ldots \vec{p}_3$

$$\vec{q}(u) = (1-u)^3 \vec{p}_0 + 3u(1-u)^2 \vec{p}_1 + 3u^2(1-u)\vec{p}_2 + u^3 \vec{p}_3$$

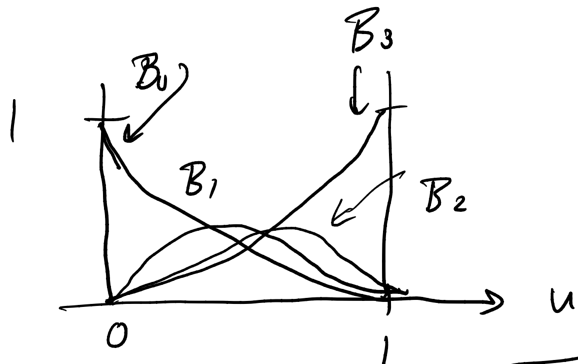


Tangencies at $u=0$, $u=1$





$$\vec{q}(u) = \underbrace{(1-u)^3}_{B_0(u)}\vec{p}_0 + \underbrace{3u(1-u)^2}_{B_1(u)}\vec{p}_1 + \underbrace{3u^2(1-u)}_{B_2(u)}\vec{p}_2 + \underbrace{u^3}_{B_3(u)}\vec{p}_3$$



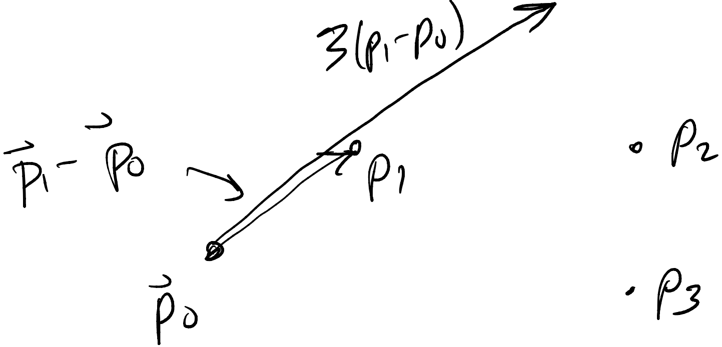
	$u=0$	$u=1$
B_0	1	0
B_1	0	0
B_2	0	0
B_3	0	1

	$u=0$	$u=1$
$B_0'(u)$	-3	0
$B_1'(u)$	3	0
$B_2'(u)$	0	-3
$B_3'(u)$	0	3

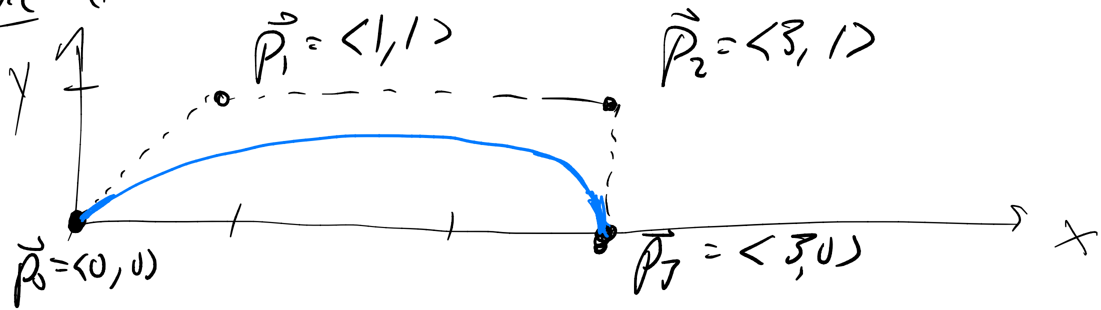
$$\boxed{\begin{array}{ll} \vec{q}(0) = \vec{p}_0 & \vec{q}\,'(0) = 3(\vec{p}_1 - \vec{p}_0) \\ \vec{q}(1) = \vec{p}_3 & \vec{q}\,'(1) = 3(\vec{p}_3 - \vec{p}_2) \end{array}}$$

Starting & Ending Points

Tangencies



<u>Example</u> (in $\mathbb{R}^2$



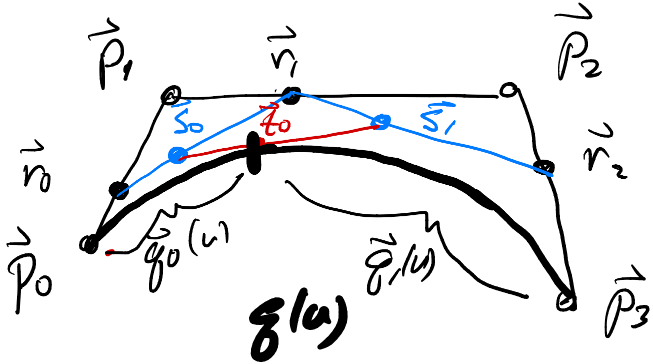
$\vec{q}(0) = \vec{p}_0$

$\vec{q}(1) = \vec{p}_3$

$\vec{q}\,'(0) = \langle 3, 3 \rangle = 3\left(\langle 1, 1 \rangle - \langle 0, 0 \rangle\right)$

$\vec{q}\,'(1) = \langle 0, -3 \rangle = 3\left(\langle 3, 0 \rangle - \langle 3, 1 \rangle\right)$

De Casteljau algorithm (For degree 3, works for other degrees)



Evaluate $\vec{q}(u)$ $\quad u \in [0,1]$

(Picture $u = 1/3$)

$\vec{r}_0 = lerp(\vec{p}_0, \vec{p}_1, u)$

$\vec{r}_1 = lerp(\vec{p}_1, \vec{p}_2, u)$

$\vec{r}_2 = lerp(\vec{p}_2, \vec{p}_3, u)$

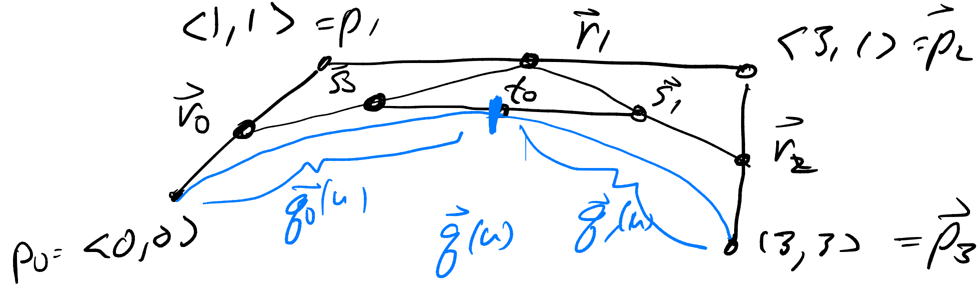
$\vec{s}_0 = lerp(\vec{r}_0, \vec{r}_1, u)$

$\vec{s}_1 = lerp(\vec{r}_1, \vec{r}_2, u)$

$\vec{t}_0 = lerp(\vec{s}_0, \vec{s}_1, u)$

Theorem t_0 is a function of u. And $t_0 = \vec{q}(u)$.

Example



$q(1/2) = ?$

$\downarrow$

$q(1/2) = \langle \frac{15}{8}, \frac{3}{4} \rangle$

$\vec{r}_0 = \langle 1/2, 1/2 \rangle$

$\vec{r}_1 = \langle 2, 1 \rangle$

$\vec{r}_2 = \langle 3, 1/2 \rangle$

$\vec{s}_0 = \langle \frac{5}{4}, \frac{3}{4} \rangle$

$\vec{s}_1 = \langle \frac{5}{2}, \frac{3}{4} \rangle$

$t_0 = \langle \frac{15}{8}, \frac{3}{4} \rangle$

<u>Theorem</u> Let $\vec{q}_0(u) = \vec{q}(\alpha u)$, for $u \in [0,1]$ Think $\alpha = \frac{1}{3}$ or $\alpha = \frac{1}{2}$

$\vec{q}_1(u) = \vec{q}((1-\alpha)u + \alpha)$, for $u \in [0,1]$

be the first α-fraction of $\vec{q}(u)$ and the $(1-\alpha)$-fraction of $\vec{q}(u)$

Then: $\vec{q}_0(u)$ is the Degree 3 Bezier curve with control points $\vec{p}_0, \vec{r}_0, \vec{s}_0, \vec{t}_0$.

and $\vec{q}_1(u)$ is the degree 3 Bezier curve with control points $\vec{t}_0, \vec{s}_1, \vec{r}_2, \vec{p}_3$

$\boxed{\text{Recursive Subdivision}}$ - Can apply repeated to a curve, until the subcurves are nearly straight.

Applications
- Robust method of calculating $\vec{q}(u)$
- Subdivide recursively to approximate by straight lines
- Exploit convex hull property.

Note For $u \in [0,1]$ $\quad B_0(u), B_1(u), B_2(u), B_3(u) \in [0,1]$

and $\quad B_0(u) + B_1(u) + B_2(u) + B_3(u) = 1$

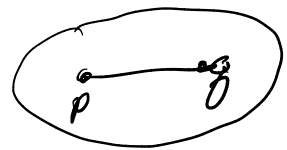
(Easy to check)

So $\vec{q}(u)$ is a weighted average of $\vec{p}_0, \vec{p}_1, \vec{p}_2, \vec{p}_3$ for $u \in [0,1]$

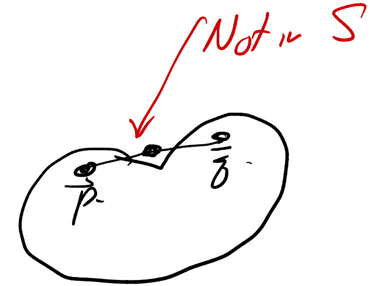
In fact $\vec{q}(u)$ is affine combination of $\vec{p}_0, \vec{p}_1, \vec{p}_2, \vec{p}_3$ for all u.

Definition A set S in $\mathbb{R}^k$ is convex if it is "closed under line segments"; i.e. if $\vec{p}, \vec{q} \in S$ and $\alpha \in [0,1]$, means $\mathrm{lerp}(\vec{p}, \vec{q}, \alpha) \in S$.

S

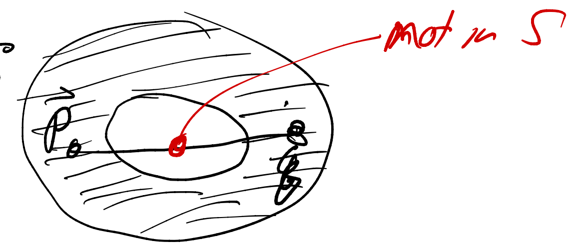


Convex



non convex

S

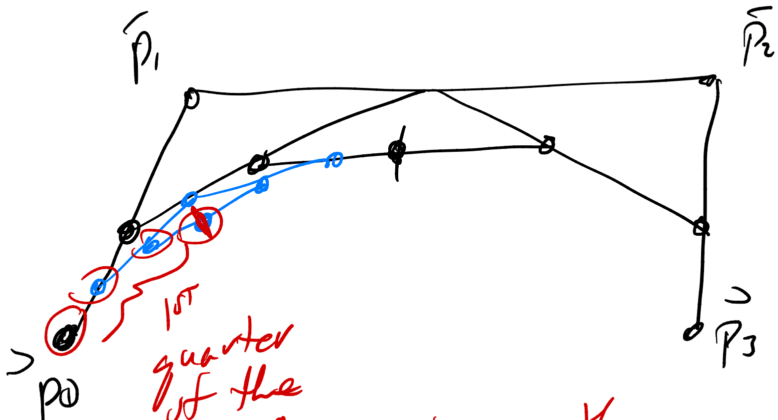


Not convex.

Theorem: If S is convex, any weighted average of members of S is also in S.

Definition The convex hull of a set S is the smallest convex set containing S.

Theorem For $u \in [0,1]$, $\vec{q}$ - a degree 3 Bezier curve $\vec{q}(u)$ is in the convex hull of its control points.



1st quarter of the curve $\vec{q}(u)$ – lies in the convex hull of the first four points circled in red

As recursively subdivide, the convex hulls of the control points get closer to lying on a line.

Bezier curves under affine transformations

Recall $\vec{q}(u)$ is an affine combination of $\vec{p}_0, \vec{p}_1, \vec{p}_2, \vec{p}_3$

Helpful principle: Let A be an affine transformation

Form $A(\vec{q}(u))$ as a new curve.

$A(q(u))$ is the degree 3 Bezier curve with control point

$$A(\vec{p}_0),\ A(\vec{p}_1),\ A(\vec{p}_2),\ A(\vec{p}_3).$$

Proof is based on: Let $\vec{x}$ be an affine combination

$\vec{x} = \sum \alpha_i \vec{y}_i$ of $\vec{y}_1 \ldots \vec{y}_k$ $\quad \alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$

Let A be an affine transformation.

$$A(\vec{x}) = A\left(\sum \alpha_i \vec{y}_i\right) = \sum \alpha_i A(\vec{y}_i)$$

($\sum \alpha_i = 1$ is important)

Proof of this last claim:

Let $A(\vec{x}) = B(\vec{x}) + \vec{t}$ $\quad$ B - is linear

$$A\left(\sum \alpha_i \vec{y}_i\right) = B\left(\sum \alpha_i \vec{y}_i\right) + \vec{t}$$

$$= B\left(\sum \alpha_i \vec{y}_i\right) + \sum \alpha_i \vec{t} \qquad \text{Since } \sum \alpha_i = 1$$

$$= \sum \alpha_i B(\vec{y}_i) + \sum \alpha_i \vec{t} \qquad B \text{ is linear}$$

$$= \sum \alpha_i \left(B(\vec{y}_i) + \vec{t}\right) = \sum \alpha_i A(\vec{y}_i)$$

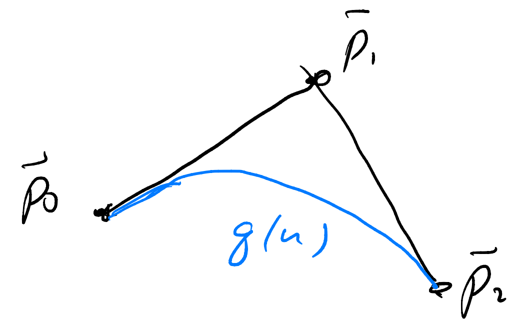
Lifting a degree 2 Bezier curve to a degree 3 Bézier curve

Recall a degree 2 Bézier curve has

$$\vec{q}(u) = (1-u)^2 \vec{p}_0 + 2u(1-u)\vec{p}_1 + u^2 \vec{p}_2$$

$$\vec{q}(0) = \vec{p}_0 \qquad \vec{q}\,'(0) = 2(\vec{p}_1 - \vec{p}_0)$$

$$\vec{q}(1) = \vec{p}_2 \qquad \vec{q}\,'(1) = 2(\vec{p}_2 - \vec{p}_1)$$



Lets find $\vec{p}_0^{\,*}, \vec{p}_1^{\,*}, \vec{p}_2^{\,*}, \vec{p}_3^{\,*}$ that define q as a degree 3 Bezier curve.

"Lifting the degree".

Use $\vec{p}_0^{\,*} = \vec{p}_0$ Since $\vec{g}(0) = \vec{p}_0$.

$\vec{p}_3^{\,*} = \vec{p}_2$ " $g(1) = \vec{p}_2$

$$\vec{p}_1^{\,*} = \vec{p}_0^{\,*} + \frac{1}{3}\left(2(\vec{p}_1 - \vec{p}_0)\right) = p_0 + \frac{2}{3}(p_1 - p_0) = lerp(\vec{p}_0, \vec{p}_1, \tfrac{2}{3})$$

$$= \frac{1}{3}\vec{p}_0 + \frac{2}{3}\vec{p}_1$$

$$\vec{p}_2^{\,\dagger} = lerp(\vec{p}_2, \vec{p}_1, \tfrac{2}{3}) = lerp(\vec{p}_1, \vec{p}_2, \tfrac{1}{3})$$