Degree 3 Bezier curve — Example

A particle starts at $\langle 0,0\rangle \in \mathbb{R}^2$ with velocity $\langle 3,0\rangle$ at time $u=0$, ends up at $\langle 3,3\rangle$ with velocity $\langle 3,3\rangle$ at time $u=1$. It is following a degree 3 curve.

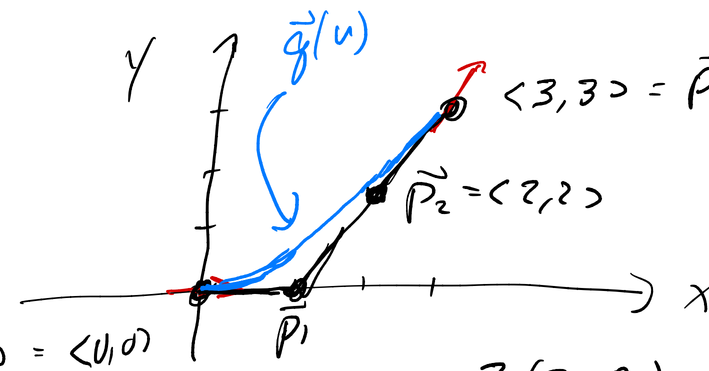
Give a degree 3 Bézier that describes the motion of the particle — By specifying its 4 control points

$\vec{p}_0 = \langle 0,0\rangle$

$\vec{p}_1 = \langle 1,0\rangle$

$\vec{p}_2 = \langle 2,2\rangle$

$\vec{p}_3 = \langle 3,3\rangle$



$p_2 = p_3 - \langle 1,1\rangle$

$3(\vec{p}_1 - \vec{p}_0) = (\text{initial velocity}) = \langle 3,0\rangle$

$3(p_3 - p_2) = (\text{final velocity})$
$= \langle 3,3\rangle$

$(p_3 - p_2) = \langle 1,1\rangle$

So $\vec{q}(u) = (1-u)^3\langle 0,0\rangle + 3u(1-u)^2\langle 1,0\rangle + 3u^2(1-u)\langle 2,2\rangle$
$+ u^3\langle 3,3\rangle.$

$$= \langle \underbrace{3u(1-u)^2 + 3u^2(1-u)2 + 3u^3}_{x(u)}, \underbrace{3u^2(1-u)\cdot 2 + 3u^3}_{y(u)} \rangle.$$

Both degree 3 polynomials.

Definition: A degree d polynomial curve in $\mathbb{R}^k$ is a function of the form

$$\vec{q}(u) = \langle q_1(u), q_2(u), \dots, q_k(u)\rangle$$

where each $q_i(u)$ is a polynomial of degree $\leq d$.

Theorem

- Every degree 2 (respectively, 3) Bezier curve is a degree 2 (respectively 3) polynomial curve.
- Conversely, every degree 2 (resp. 3) polynomial curve is a degree 2 (resp. 3) Bezier curve.

Example Define $\vec{g}(u) = \langle 2u, 4u^2 \rangle$, for $u \in [0,1]$.

$\vec{g}(0) = \langle 0,0 \rangle$

$\vec{g}(1) = \langle 2,4 \rangle$

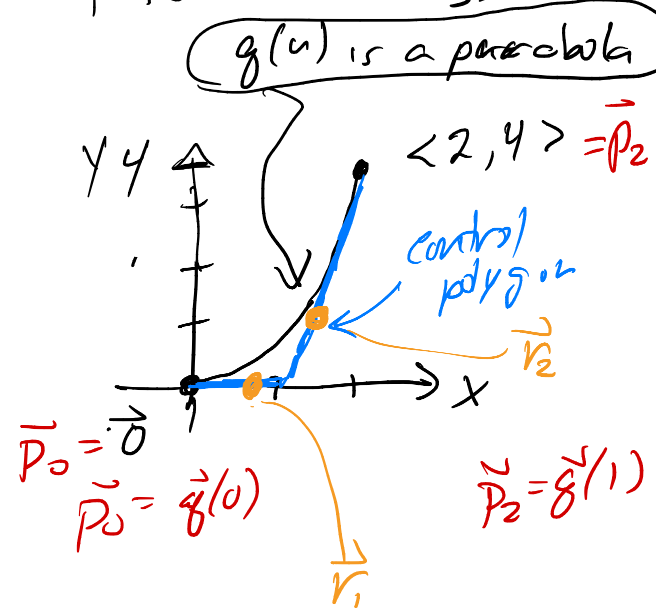
$g'(u) = \langle 2, 8u \rangle$

$g'(0) = \langle 2, 0 \rangle$

$g'(1) = \langle 2, 8 \rangle$

Problem

Express $\vec{g}(u)$ as a degree 2 Bezier curve



$$\left.\begin{aligned} 2(\vec{p}_1 - \vec{p}_0) &= \vec{g}\,'(0) \\ 2(\vec{p}_2 - \vec{p}_1) &= \vec{g}\,'(1) \end{aligned}\right\} \quad \vec{p}_1 = \langle 1, 0 \rangle$$

$p_1 - p_2 = \langle 1, 0 \rangle$

$p_3 - p_2 = \langle 1, 4 \rangle$

$\vec{p}_1$ is the unique point at the intersection of the tangent lines

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Continuing the example - express $\vec{g}(u)$ as a degree 3 Bezier curve, with control points $\vec{r}_0, \vec{r}_1, \vec{r}_2, \vec{r}_3$

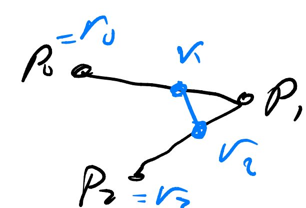
To lift a degree 2 Bezier to a degree 3 Bezier curve

$\vec{r}_0 = \vec{p}_0$ $\qquad \vec{r}_1 = \mathrm{lerp}(\vec{p}_0, \vec{p}_1, 2/3)$

$\vec{r}_3 = \vec{p}_2$ $\qquad \vec{r}_2 = \mathrm{lerp}(\vec{p}_2, \vec{p}_1, 2/3) = \mathrm{lerp}(\vec{p}_1, \vec{p}_2, 1/3)$

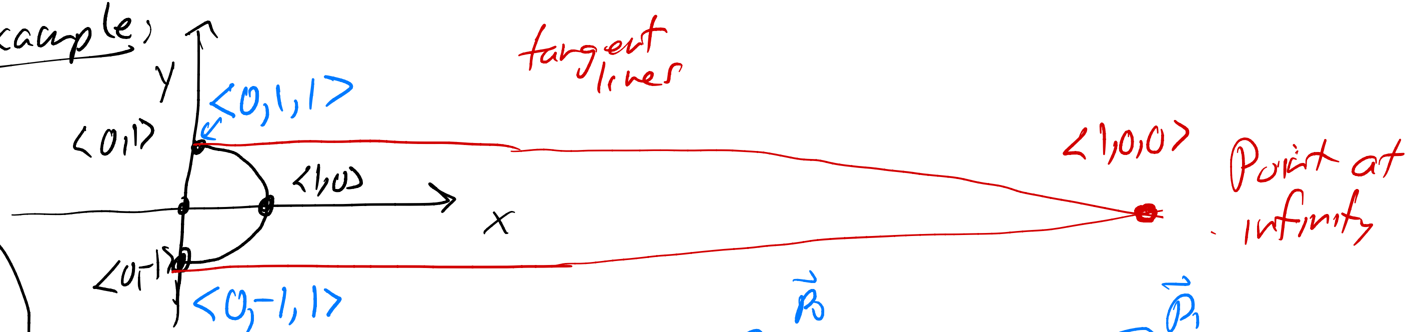
$\vec{r}_1 = \frac{2}{3}\langle 1, 0 \rangle = \langle \frac{2}{3}, 0 \rangle$

$\vec{r}_1 = \frac{1}{3}\langle 2, 4 \rangle + \frac{2}{3}\langle 1, 0 \rangle = \langle \frac{4}{3}, \frac{4}{3} \rangle$



Drawing circular arcs w/ degree 2 Bézier on homogeneous coordinates

Example:



Right half of a unit circle

Verify: Let $\vec{g}(u) = (1-u)^2 \underbrace{\langle 0,1,1\rangle}_{\vec{p}_0} + 2u(1-u)\underbrace{\langle 1,0,0\rangle}_{\vec{p}_1} + u^2 \underbrace{\langle 0,-1,1\rangle}_{\vec{p}_2}$

So $\vec{g}(u) = \langle \underbrace{2u(1-u)}_{x(u)}, \underbrace{(1-u)^2 - u^2}_{y(u)}, \underbrace{(1-u)^2 + u^2}_{z(u)} \rangle$

↳ Represents $\left\langle \frac{x(u)}{z(u)}, \frac{y(u)}{z(u)} \right\rangle \in \mathbb{R}^2$

Easy to check $\left(\frac{x(u)}{z(u)}\right)^2 + \left(\frac{y(u)}{z(u)}\right)^2 = 1$

i.e. $x(u)^2 + y(u)^2 = z(u)^2$

So $\vec{g}(u)$ always lies on the unit circle.

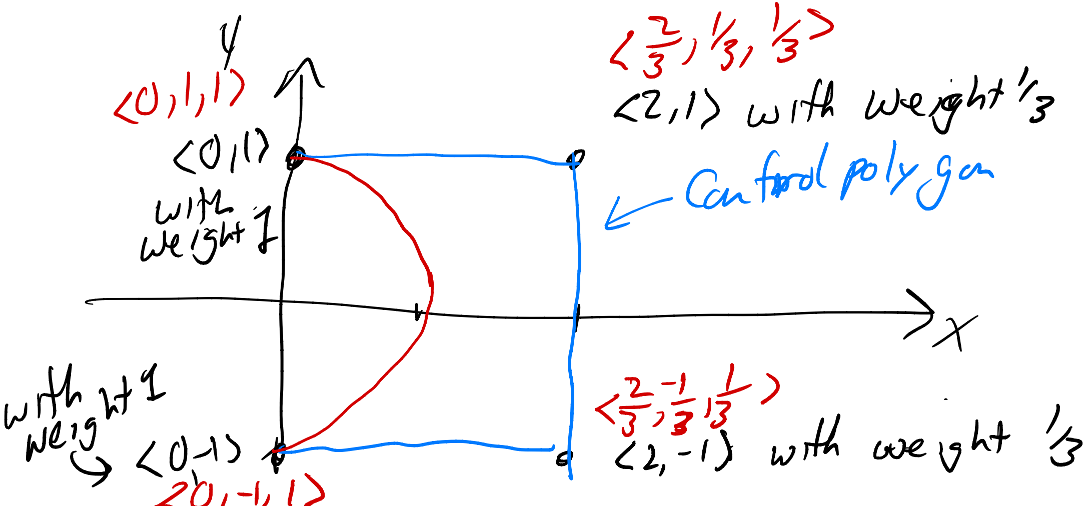
Lift to a degree 3 Bezier curve, with control points $\vec{r}_0, \vec{r}_1, \vec{r}_2, \vec{r}_3$

$\vec{r}_0 = \vec{p}_0 \quad \vec{r}_3 = \vec{p}_2 \qquad \vec{r}_1 = \mathrm{lerp}(p_0, p_1, 2/3) =$

$= \frac{1}{3}\langle 0,1,1\rangle + \frac{2}{3}\langle 1,0,0\rangle$

$= \langle \frac{2}{3}, \frac{1}{3}, \frac{1}{3}\rangle$ represents $\langle 2,1\rangle \in \mathbb{R}^2$

Likewise $\vec{r}_2 = \mathrm{lerp}(p_2, p_1, 2/3) = \langle \frac{2}{3}, \frac{-1}{3}, \frac{1}{3}\rangle$, represents $\langle 2,-1\rangle$



Why is $\langle 1,0,0\rangle$ a point at infinity?

Think of the limit of

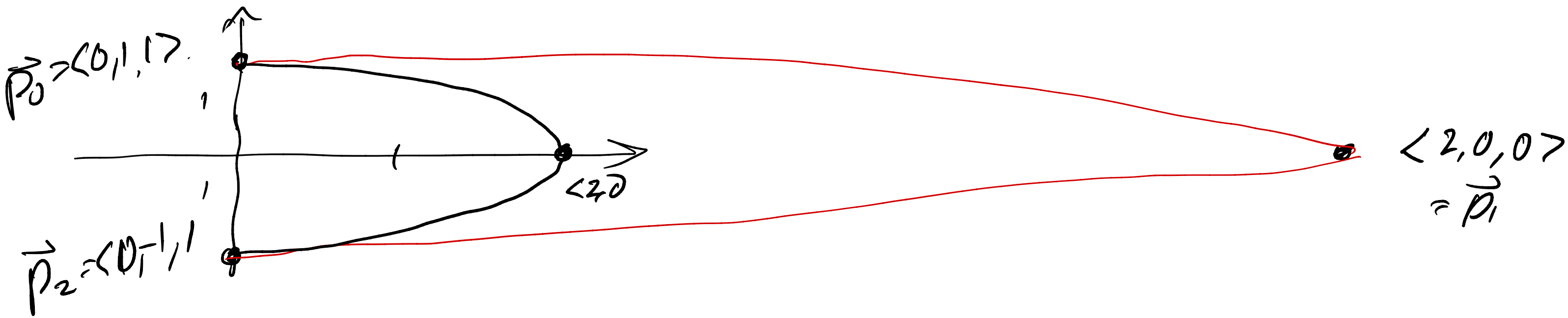
$$\langle 1,0,1\rangle,\ \langle 1,0,\tfrac{1}{2}\rangle,\ \langle 1,0,\tfrac{1}{3}\rangle,\ \langle 1,0,\tfrac{1}{4}\rangle \ldots\ \langle 1,0,\tfrac{1}{n}\rangle \ldots$$

seems to converge to $\langle 1,0,0\rangle$

they represent $\langle 1,0\rangle,\ \langle 2,0\rangle,\ \langle 3,0\rangle,\ \langle 4,0\rangle \ldots\ \langle n,0\rangle \ldots$

which could reasonably be said to converge to a point at infinity.

(Note $\langle -1,0,0\rangle$ is the somehow a different of the same point at infinity ;)



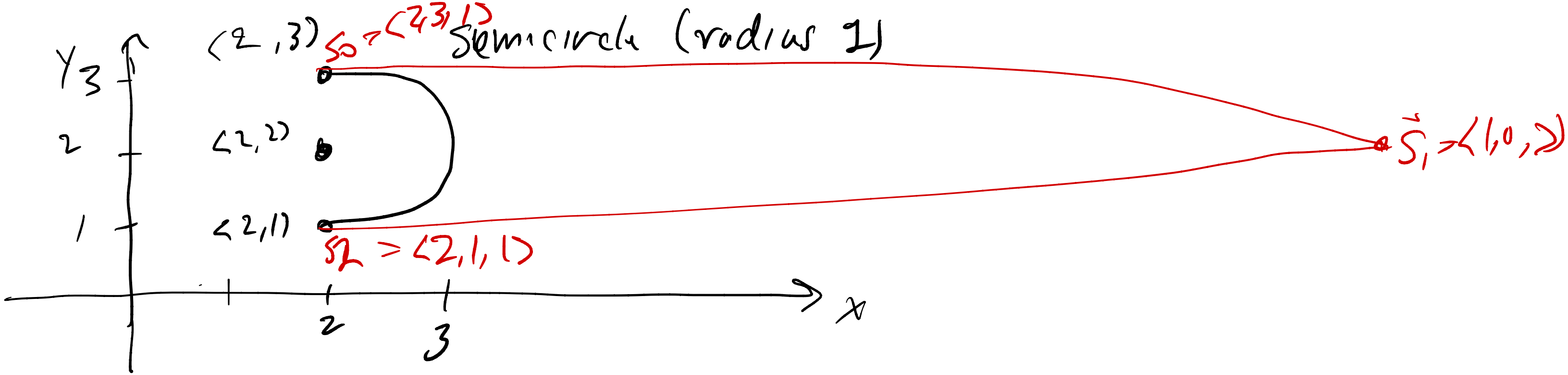
this $\vec{p}_0, \vec{p}_1, \vec{p}_2$ defines an ellipse of radius 2 in x-direction and radius 1 in the y-direction.

Multiplying the original semi circle by the affine transform matrix

$$S_{\langle 2,1,1 \rangle} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

old control points $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$



What the control points for this semicircle?

<u>Translation</u>

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = T_{(2,2)}$$

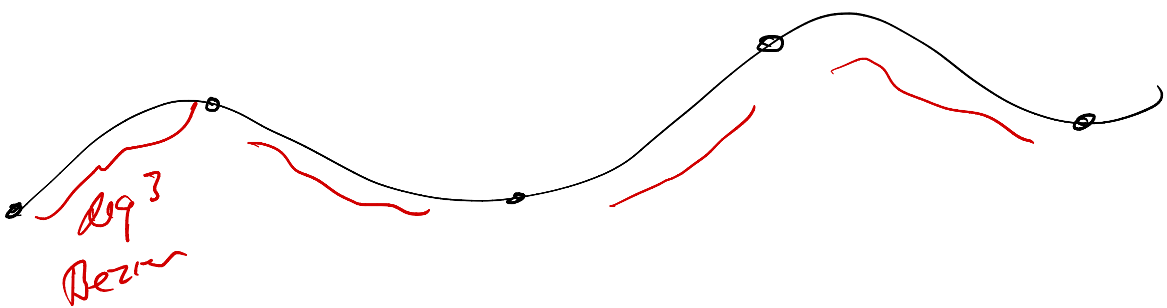
$$T_{(2,2)} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \vec{s}_0 \qquad T_{(2,2)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{s}_1$$

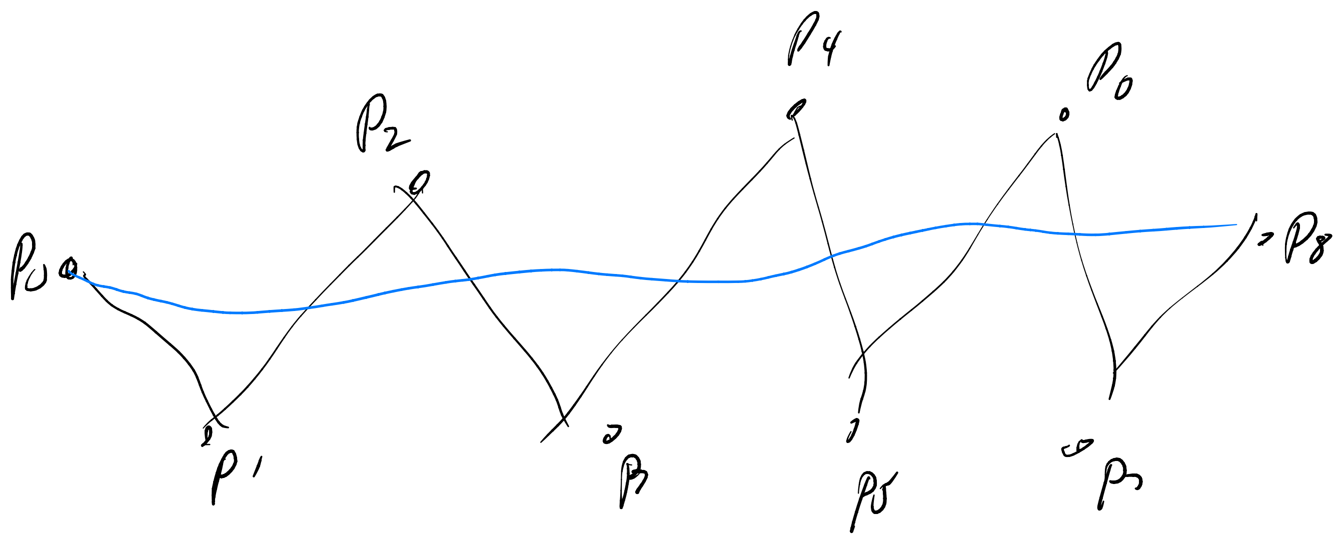
$$T_{(2,2)} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \vec{s}_2$$

Similar constructions work for all conic sections

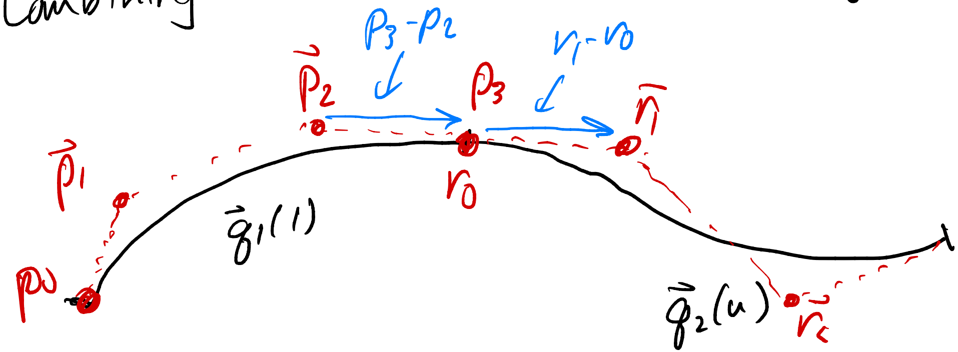
using rational Bezier curves (rational splines)

Piece-wise Bezier curves





Combining 2 curves to make a single



2 common goals:

① Curve is continuous.

$$\text{Let } \vec{q}(u) = \begin{cases} q_1(u) & \text{if } u \in [0,1] \\ q_2(u-1) & \text{if } u \in [1,2] \end{cases}$$

So $\vec{q}(u)$ has domain $[0,2]$.

Need $\vec{p}_3 = \vec{r}_0$ for continuity

② C^1-continuity continuous 1^{ST}-derivativity

Need $\bar{q}_1'(1) = \bar{q}_2'(0)$

$$3(\vec{p}_3 - \vec{p}_2) = 3(\vec{r}_1 - \vec{r}_0)$$

$$\vec{p}_3 - \vec{p}_2 = \vec{r}_1 - \vec{r}_0$$

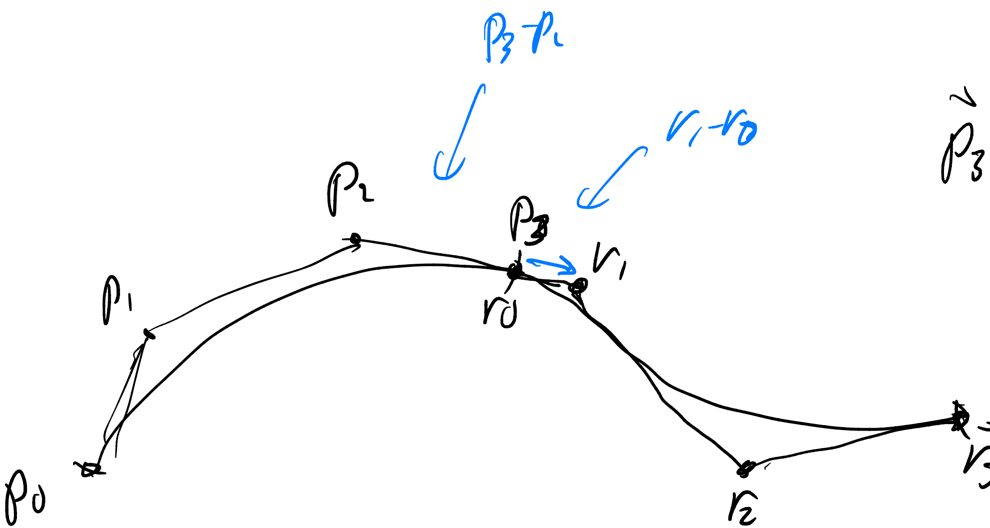
$\bar{r}_0 = \bar{p}_3$ is the midpoint of $\bar{p}_2$ and $\bar{r}_1$

or (2') G^1-continuity - Geometric continuity (of 1^{st} derivative)

Ending slope of q_1 is equal to the starting slope of $\vec{q}_2$

For this $\boxed{\vec{p}_3 - \vec{p}_2 = \alpha \cdot (\vec{r}_1 - \vec{r}_0) \text{ for some } \alpha > 0.}$

is sufficient



$\vec{p_3} - \vec{p_2}$, $\vec{r_1} - \vec{r_0}$ are in the same direction.

G^1-continuity.