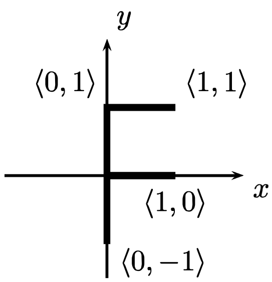
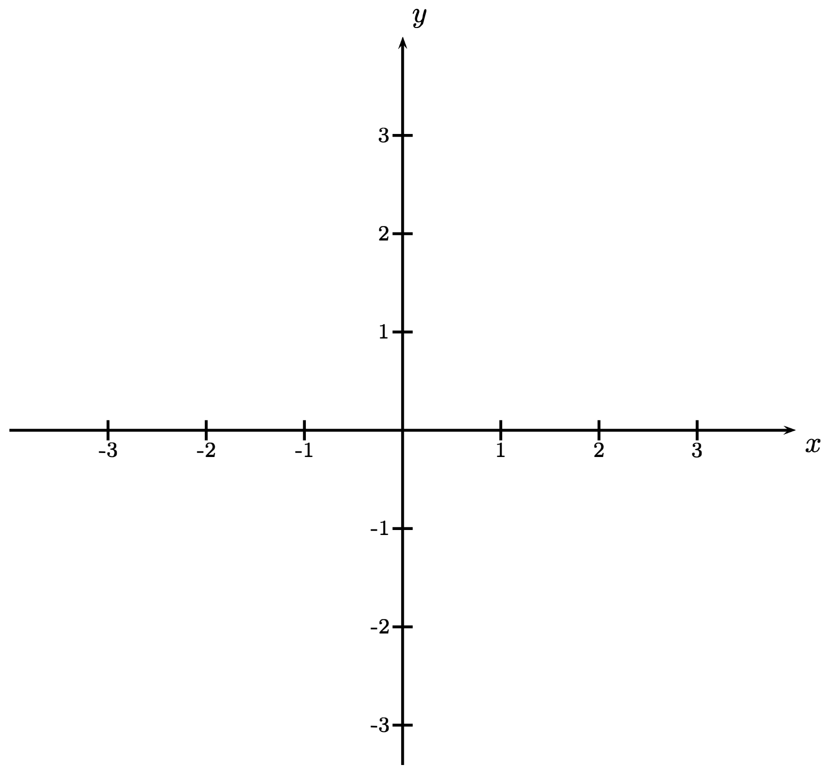
1. [20 points] Consider the following 3×3 matrix M representing a transformation in $\mathbb{R}^2$ over homogeneous coordinates.

a. Draw the image of the "F" under this transformation on the large axes to the right. Be sure to label enough points to make your answer clear.

$$M = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



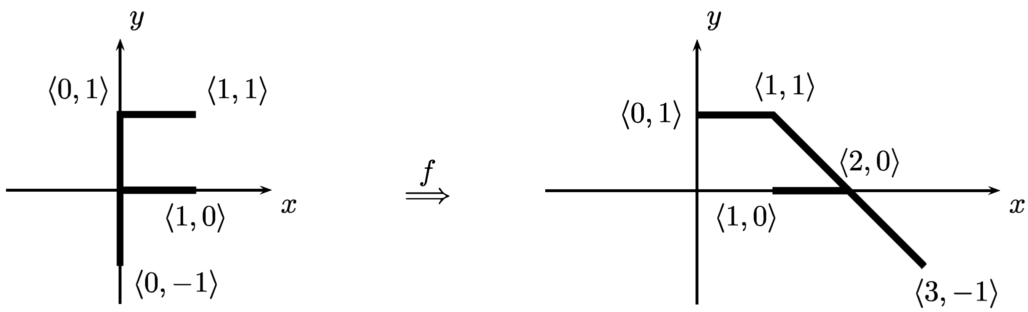


b. Is this transformation rigid? Is it orientation preserving?

c. Express the transformation as a generalized rotation $R_\varphi^{\mathbf{v}}$ by giving $\mathbf{v}$ and φ — or explain why this is not possible. (Recall that a generalized rotation $R_\varphi^{\mathbf{v}}$ is a rotation around $\mathbf{v}$.)

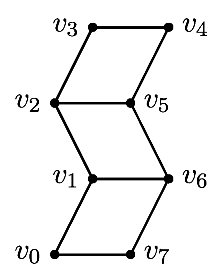
d. Express the transformation as a composition of zero or more rotations R_θ, scalings $S_{\mathbf{u}}$, and translations $T_{\mathbf{u}}$. (Do not give matrices.)

2. [10 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.



Give a 3×3 matrix which represents f over homogeneous coordinates.

3. [10 points] Eight vertices are used to specify the three quadrilaterals as shown. List the vertices in the correct order to render the quadrilaterals as a single triangle strip. List the vertices so that the faces are facing towards the viewer as shown (when using the default conventions for front faces).



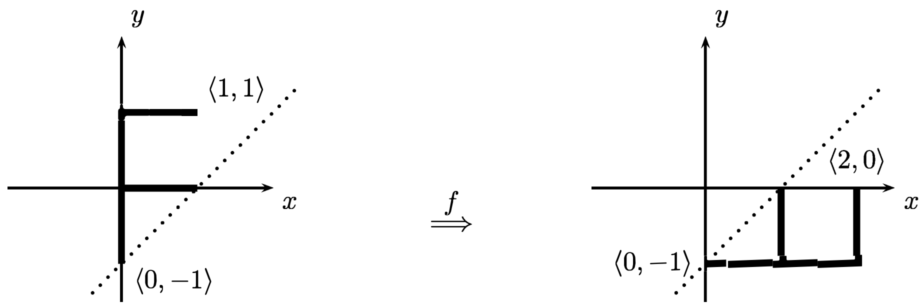
6. [10 points] State the definition of "affine transformation".

7. [10 points] Describe the depth buffer method for hidden surfaces. What are its advantages? What are its disadvantages?

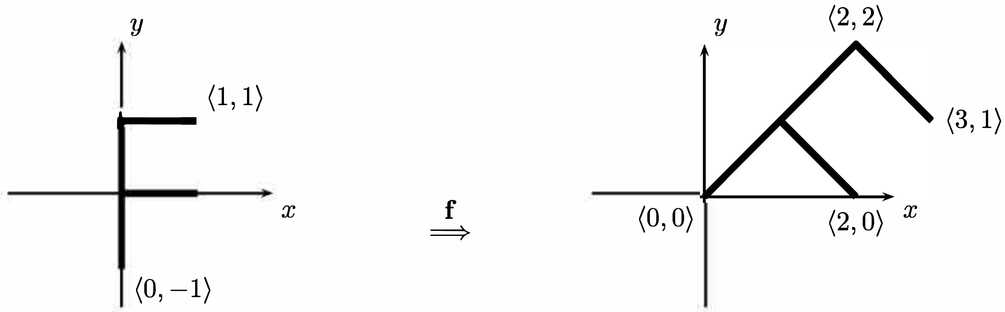
3. [10 points] Give the definition of **linear transformation**.

4. [20 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be the transformation that reflects points across the line $y = x - 1$. In particular, it maps a "F" as shown in the picture below.

Give the matrix that represents f over homogeneous coordinates.



1. [20 points] Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined to be the affine transformation that maps an "F" as shown in the picture below.



a. Is f a rigid transformation? Explain why or why not.

b. Express f in the form $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$ with M a 2×2 matrix.

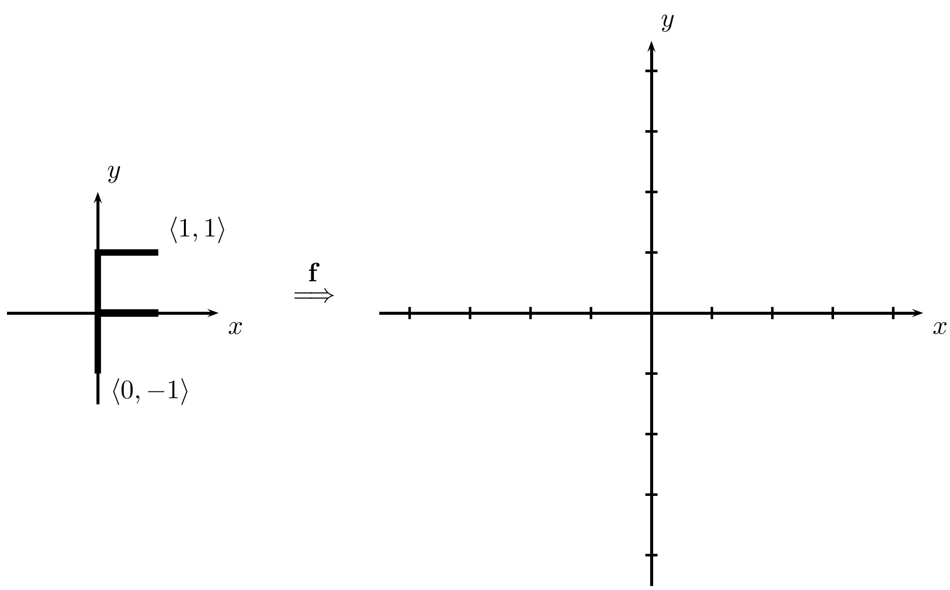
c. ~~Give a sequence of "pseudo" OpenGL commands that will draw the "F" in the position shown on the right. Use commands such as `drawF()` (draws "F" in the position shown on the left), `pglRotatef(···)`, `glTranslate2f(···)`, `glLoadIdentity()`, and `pglScale2f(···)`.~~

Express the transformation f as a composition of transformations of the forms R_θ, $T_{\vec{u}}$, and $S_{\langle a,b \rangle}$.

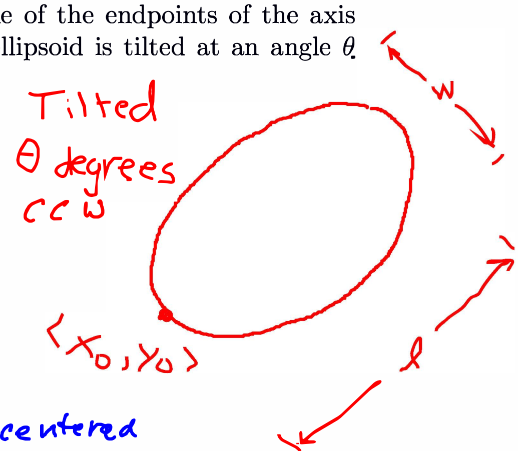
2. [15 points] Consider the following 3×3 matrix M that operates on the homogeneous coordinates of points in $\mathbb{R}^2$.

$$\begin{pmatrix} -2 & -2 & 2 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In the empty graph on the right, draw the image of the "F" under the affine map on $\mathbb{R}^2$ that is defined by the matrix M. Draw to scale, and label points as needed.



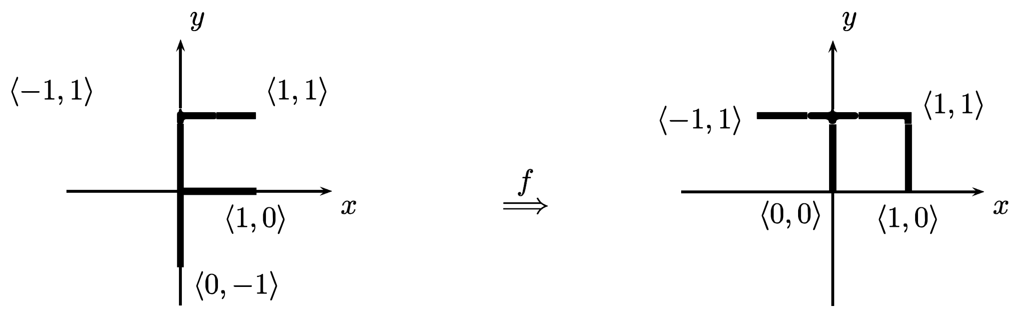
1. [20 points] This problem concerns transformations in $\mathbb{R}^2$. Suppose you are given a function `DrawCircle()` that draws a unit circle centered at the origin (radius equals one). Give a code fragment that will draw an ellipse as shown in the figure. The length of the ellipse is ℓ and the width is w. One endpoint of the ellipse is at $\langle x_0, y_0 \rangle$ in $\mathbb{R}^2$, namely, one of the endpoints of the axis along which the length ℓ is measured. The ellipsoid is tilted at an angle θ (measured in degrees).



Your code fragment that draws the ellipse may use any of the following pseudo OpenGL commands. `glMatrixMode()`, `glLoadIdentity()`, `pglRotatef()`, `pglTranslatef()`, `pglLoadMatrix()`, `pglMultMatrix()`, `pglScalef()`, and `DrawCircle()`.

Decribe the 3x3 matrix which will transform the unit circle centered at the origin to be ellipse as pictured. Describe the matrix as a composition of rotations R_θ, translations $T_{\vec{u}}$ and scalings $S_{\langle a,b \rangle}$.

1. [36 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.



a. Express f in the form $f(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$ with M a 2×2 matrix.

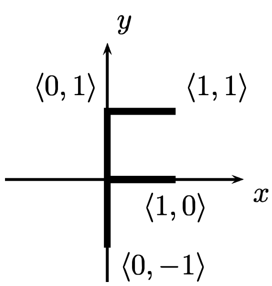
e. Now consider the inverse f^{-1} of the transformation f. Give a 3×3 matrix N that represents f^{-1} in homogeneous coordinates.

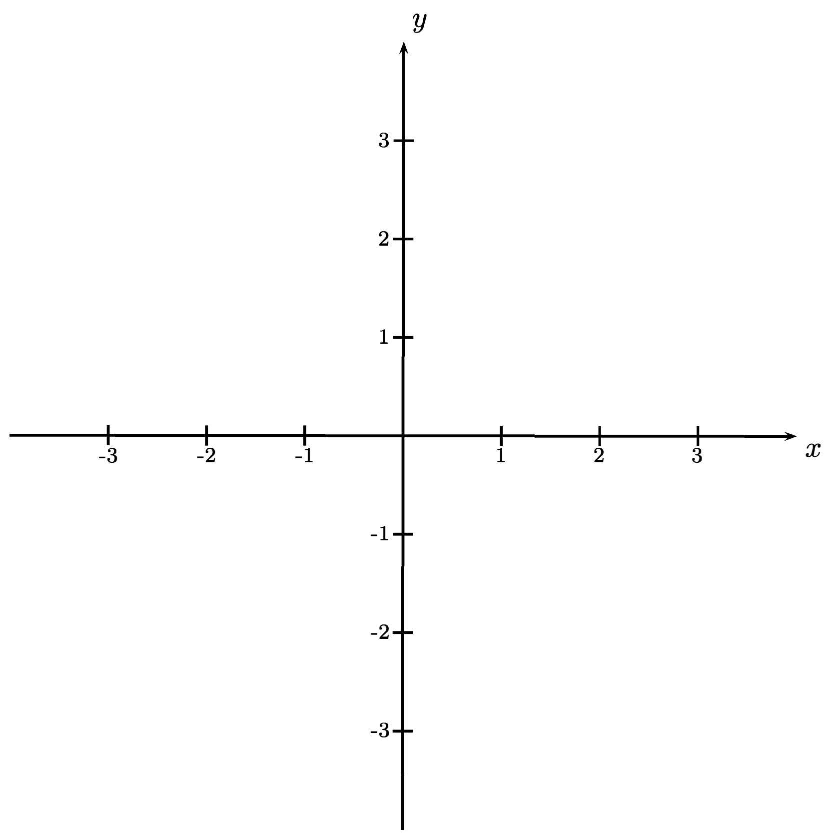
f. Express f as a generalized rotation $f = R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ by giving the rotation angle θ and the center point $\mathbf{u}$ of the generalized rotation, or explain why this is not possible

1. [10 points] Recall that a generalized rotation $R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ is the rigid orientation-preserving transformation which rotates counterclockwise around the point $\mathbf{u}$ (holding the point $\mathbf{u}$ fixed. Express $R_\theta^{\mathbf{u}}$ as a composition of rotations R_φ and translations $T_{\mathbf{v}}$.

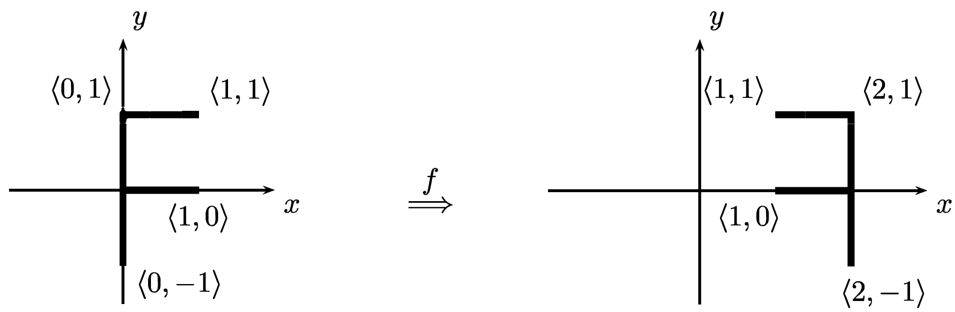
2. [10 points] Consider the following 3×3 matrix M representing a transformation in $\mathbb{R}^2$ over homogeneous coordinates. (Watch out for the lower right entry!) Draw the image of the "F" under this transformation on the large axes to the right. Be sure to label enough points to make your answer clear.

$$M = \begin{pmatrix} 0 & 2 & -2 \\ -4 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$





3. [40 points] Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that maps an "F" as shown in the picture below.



a. Give a 3×3 matrix which represents f over homogeneous coordinates.

b. Now consider the inverse f^{-1} of the transformation f. Give a 3×3 matrix N that represents f^{-1} in homogeneous coordinates.

c. Express f as a composition of rotations R_θ, translations $T_{\mathbf{u}}$, and/or scalings $S_{\langle a,b\rangle}$, or explain why this is not possible.

d. Express f as a generalized rotation $f = R_\theta^{\mathbf{u}}$ in $\mathbb{R}^2$ by giving the rotation angle θ and the center point $\mathbf{u}$ of the generalized rotation, or explain why this is not possible

1. [20 points] Consider the following OpenGL commands:

```
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glRotatef( 90.0, 0.0, 1.0, 0.0 );
glTranslatef( 2.0, 0.0, 0.0 );
glScalef( 2.0, 1.0, 1.0 );
```

What will the 4 × 4 modelview matrix be equal to after these commands have executed?

Consider the transformation

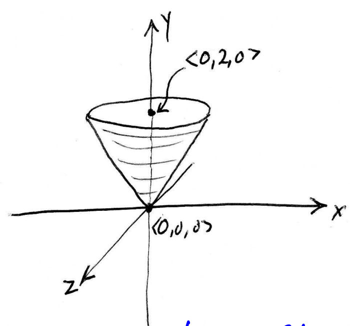
$$R_{\frac{\pi}{2}, \langle 0,1,0 \rangle} \circ T_{\langle 2,0,0 \rangle} \circ S_{\langle 2,1,1 \rangle}$$

Give the 4x4 matrix which represents this transformation over homogeneous coordinates.

1. [20 points] This problem concerns transformations in $\mathbb{R}^3$. Suppose you are given a function `DrawCone()` that draws a cone of height 1, and base radius 1. This cone drawn by `DrawCone()` is situated centered around the y-axis with its base on the xz plane and the tip of the cone at $\langle 0, 1, 0 \rangle$.

a. Give a ~~code fragment~~ description of a transformation that will draw the cone as shown in the figure: the cone is to be drawn upside down, and with height 2 and base radius 2. Its tip is now at the origin; it is still centered around the y-axis.

~~Your code fragment that draws the cone may use any of the following pseudo OpenGL commands: `glMatrixMode()`, `glLoadIdentity()`, `glRotatef()`, `glTranslatef()`, `glLoadMatrix()`, `glMultMatrix()`, `glScalef()`, and `DrawCone()`.~~



Your description of the matrix may be given as a composition of rotations ($R_{\theta,\hat{u}}$), translations ($T_{\vec{u}}$) and scalings ($S_{\langle a,b,c \rangle}$).

b. Give a 4×4 homogeneous matrix that gives the same transformation as is used in your answer for part a.

3. [12 points] Suppose the function `drawTwoPoints()` draws a point at $\langle 0, 0, 0\rangle$ and another point at $\langle 1, 1, 0\rangle$.

a. Consider the ~~sequence of OpenGL commands:~~ transformation A defin

```
glLoadIdentity();
glTranslatef(-1, 0, 0);
glScalef(2, 1, 2);
glRotatef(90, 0, 1, 0);
drawTwoPoints();
```

$$T_{\langle -1,0,0\rangle} \circ S_{\langle 2,1,2\rangle} \circ R_{\frac{\pi}{2},\langle 0,1,0\rangle}$$

When the drawTwoPoints() is called, where does the point it draws at $\langle 0, 0, 0\rangle$ actually get placed (as transformed by the ModelView matrix)? And, where does the point it draws at $\langle 1, 1, 0\rangle$ get placed?

When transformed by A, where do these two points lie?

b. Now consider the slightly different ~~sequence of OpenGL commands:~~ transformation B:

```
glLoadIdentity();
glRotatef(90, 0, 1, 0);
glScalef(2, 1, 2);
glTranslatef(-1, 0, 0);
drawTwoPoints();
```

$$R_{\frac{\pi}{2},\langle 0,1,0\rangle} \circ S_{\langle 2,1,2\rangle} \circ T_{\langle -1,0,0\rangle}$$

~~When the drawTwoPoints() is called, where does the point it draws at $\langle 0, 0, 0\rangle$ actually now get placed? (You only need to answer about this one point.)~~

Where are the two points placed by B?

4. [10 points] Suppose we are modelling a Solar System and need to define a transformation A which will place the Earth in the right position. The sun is at the origin. The Earth is distance d from the Sun, lying in the xz plane. We wish to revolve the Earth by angle θ around the Sun (for the time of year). We wish to rotate the Earth on its axis by angle φ (for time of day). We wish to draw the Earth with radius r. (There is no tilt!)

Suppose we have a routine that draws the Earth as a radius 1 sphere centered at the origin. What transformation A needs to be used to place the Earth as desired? Express your answer A as a composition of rotations $R_{\psi,\vec{u}}$. translations $T_{\mathbf{u}}$, and scalings $S_{a,b,c}$.

3. [20 points] A light source is placed at the origin in $\mathbb{R}^3$, and it casts shadows onto the plane defined by $z = -10$. Thus, the plane is like an infinite wall parallel to the xy-plane, placed at $z = -10$.

For $\mathbf{x} = \langle x_1, y_1, z_1 \rangle$ a point in $\mathbb{R}^3$ where $z_1 < 0$, let $A(\mathbf{x}) = \langle x_2, y_2, z_2 \rangle$ be the point on the wall where the shadow of $\mathbf{x}$ is. This means that $z_2 = -10$. Give a 4×4 matrix that represents the transformation A over homogenous coordinates, **or**, prove that there is no such matrix.

5. [20 points] A light source is placed at $\langle -10, 0, 0 \rangle$ and it casts shadows onto the yz-plane P defined by $x = 0$. The yz-plane is like an infinite wall.

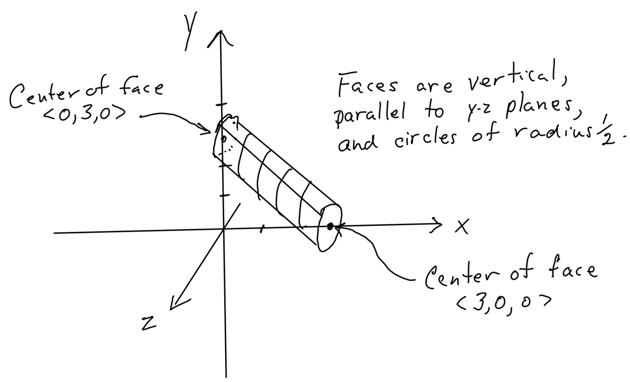
When $\langle x, y, z \rangle$ is a point in $\mathbb{R}^3$ with $-10 < x \leqslant 0$, define $A(\langle x, y, z \rangle)$ to be the position of the shadow of the point on the yz-plane. For example, $A(\langle -5, 1, 2 \rangle) = \langle 0, 2, 4 \rangle$, and $A(\langle -8, 1, 2 \rangle) = \langle 0, 5, 10 \rangle$

a. Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $A(\langle x, y, z \rangle) = \langle x', y', z' \rangle$. That is, give formulas for x', y', z' in terms of x, y, z.

b. Give a 4×4-matrix that represents the transformation A over homogeneous coordinates.

4. [20 points] Suppose $\mathcal{C}$ is a radius 1, height 2 cylinder centered at the origin, with central axis the y-axis. The top face of $\mathcal{C}$ is the horizontal disk of radius one centered $\langle 0, 1, 0\rangle$. The bottom face of $\mathcal{C}$ is the horizontal disk of radius one centered $\langle 0, -1, 0\rangle$. ("Horizontal" means parallel to the xz-plane.)

Let $\mathcal{D}$ be the skewed cylinder shown in the figure which has central axis the line containing $\langle 3, 0, 0\rangle$ and $\langle 0, 3, 0\rangle$. The right face of $\mathcal{D}$ is the vertical radius $\frac{1}{2}$ disk, parallel to the yz plane centered at $\langle 3, 0, 0\rangle$ and the left face of $\mathcal{D}$ the radius 1 disk lying in the yz-plane centered at $\langle 0, 3, 0\rangle$.



Give a 4×4 matrix M which transforms $\mathcal{C}$ to $\mathcal{D}$. For full credit, keep the outward faces still outward facing after the transformation. (In other words, M is orientation preserving and does not turn the cylinder inside out.)

5. [20 points] A light source is placed at origin and it casts shadows onto the plane P defined by $x = 20$. This plane is like an infinite wall, parallel to the yz-plane.

When $\langle x, y, z\rangle$ is a point in $\mathbb{R}^3$ with $0{<}x{\leqslant}20$, define $A(\langle x, y, z\rangle)$ to be the position of the shadow of the point on the plane P. For example, $A(\langle 5, 1, 2\rangle) = \langle 20, 4, 8\rangle$, and $A(\langle 2, 1, 2\rangle) = \langle 20, 10, 20\rangle$

a. Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $A(\langle x, y, z\rangle) = \langle x', y', z'\rangle$. That is, give formulas for x', y', z' in terms of x, y, z.

b. Give a 4×4 matrix that represents the transformation A over homogeneous coordinates.

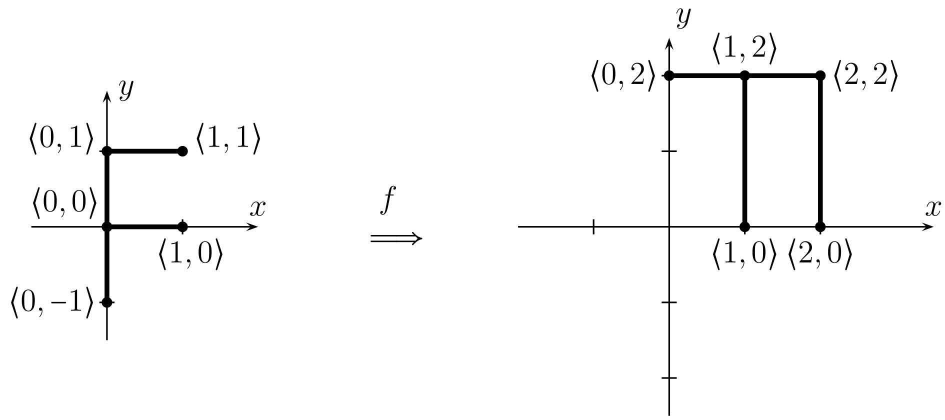
Question 1: Briefly describe the Painter's algorithm. Be sure to include comments on

a. The purpose of the Painter's algorithm.

b. How the Painter's algorithm works.

c. Disadvantages of the Painter's algorithm.

Question 3: An affine transformation f of $\mathbb{R}^2$ maps the standard "F" shape as shown:



The same transformation f is used in Question 4 on the next page.

Express f as a composition of translations $T_{\mathbf{u}}$, scalings $S_{\alpha,\beta}$, and rotations R_θ.

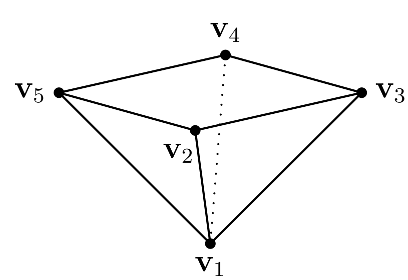
Question 4: Continue to work with the same transformation f as in Question 3. Answer (a) and (b) below.

(a) Give a 3×3 matrix that represents f over homogeneous coordinates.

(b) Give a 3×3 matrix that represents f^{-1} over homogeneous coordinates.

Question 2: A "square cone" (or, an upside-down four-sided pyramid) has a flat, square top, and four triangular sides. Its bottom vertex is $\mathbf{v}_1 = \langle 0, 0, 0 \rangle$. Its four top vertices are

$\mathbf{v}_2 = \langle 0, 2, 2 \rangle$ (front);
$\mathbf{v}_3 = \langle 2, 2, 0 \rangle$ (right);
$\mathbf{v}_4 = \langle 0, 2, -2 \rangle$ (back); and
$\mathbf{v}_5 = \langle -2, 2, 0 \rangle$ (left).



A triangle fan is used to render the four bottom triangles of the square cone. Give the vertices — in a correct order — used for the triangle fan. How many vertices should be used in the list of vertices for the triangle fan? List the vertices in an order that makes the front faces facing outward (downward) from the cone.

Question 1: (Homogeneous coordinates in $\mathbb{R}^3$.)

a. What point in $\mathbb{R}^3$ is represented by the homogeneous coordinates $\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6} \rangle$?

b. Give three different homogenous representations for the the point $\langle 1, -2, 3 \rangle$.

Question 2: Let $\mathbf{u} = \langle 1, 0, 2 \rangle$. Define $f(\mathbf{x}) = \mathbf{x} \times \mathbf{u}$ ("$\times$" is vector cross product.)

a. Is f a linear map?

b. Is f an affine map?

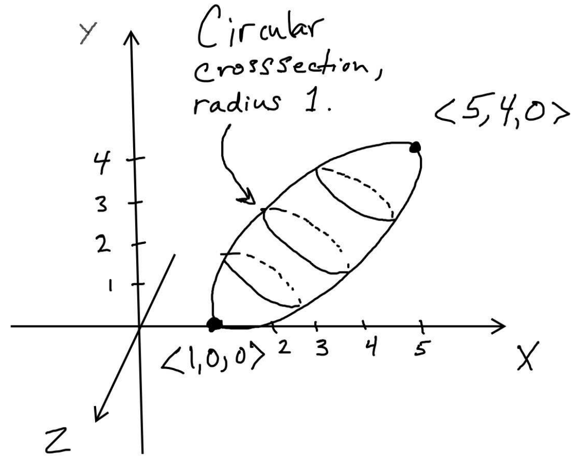
c. Give a 4×4 matrix that represents f over homogeneous coordinates.

Question 5: A light source is placed at $\langle 10, 0, 0 \rangle$ and it casts shadows onto the plane P defined by $x = 2$. Note that P is parallel to the yz plane, and acts like an infinite wall.

When $\langle x, y, z \rangle$ is a point in $\mathbb{R}^3$ with $2 \le x < 10$, define $h(\langle x, y, z \rangle)$ to be the position of the shadow of the point on the yz-plane. For example, $h(\langle 4, 3, -6 \rangle) = \langle 2, 4, -8 \rangle$.

a. Working in ordinary coordinates (not homogeneous) give the formula expressing the mapping $h(\langle x, y, z \rangle) = \langle x', y', z' \rangle$. That is, give formulas for x', y', z' in terms of x, y, z.

For question 3 and 4: Let $\mathcal{S}$ be the unit sphere centered at the origin. Consider an ellipsoid $\mathcal{E}$ which has circular crosssection of radius 1, and its major axis has one end at $\langle 1, 0, 0 \rangle$ and the other end at the point $\langle 5, 4, 0 \rangle$. We wish to define an affine transformation g that maps the unit sphere $\mathcal{S}$ to the ellipsoid $\mathcal{E}$. For full credit, please give answers that make g be orientation preserving.



Question 3: Express g as a composition of rotations $R_{\theta,\mathbf{u}}$, scalings $S_{\langle a,b,c \rangle}$ and translations $T_{\mathbf{u}}$. (There are many possible answers: you only need to give one!)

Question 4: Give a 4×4 matrix that represents g over homogeneous coordinates. (There are again multiple possible answers; your answer for b. does not need to correspond to your answer for a.

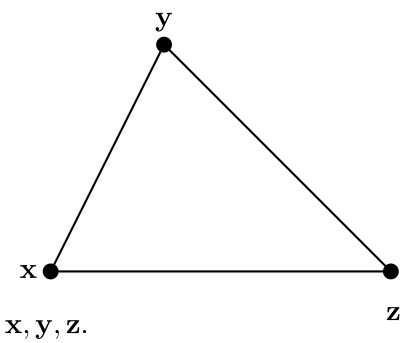
1. [20 points] Let $\mathbf{x} = \langle 0, 2\rangle$ and $\mathbf{y} = \langle 3, 6\rangle$.

(a) What is $\text{LERP}(\mathbf{x}, \mathbf{y}, \frac{1}{3})$?

(b) What is $\text{LERP}(\mathbf{y}, \mathbf{x}, -\frac{1}{3})$? (Watch the order of the parameters!)

(c) Let L be the line containing $\mathbf{x}$ and $\mathbf{y}$. What point on L is closest to $\langle 2, 3\rangle$?

2.[20 points] Let $\mathbf{x} = \langle -1, 0\rangle$, $\mathbf{y} = \langle 0, 2\rangle$, and $\mathbf{z} = \langle 2, 0\rangle$.



(a) What point $\mathbf{u}$ has barycentric coordinates $\langle \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\rangle$ with respect to these three points? Give $\mathbf{u}$ explcitly, and draw its approximate location on the figure.

(b) Let $\mathbf{v} = \langle 0, 1\rangle$. Express $\mathbf{v}$ as an explicit weighted average of $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

3. [6 points] State the definition of an "affine combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$".

4. [14 points] The Phong lighting calculations use three unit vectors ℓ, $\mathbf{n}$ and $\mathbf{v}$ in addition to the material and light properties. The Phong calculation for spectral light, involves forming either a dot product $\mathbf{r}\cdot\mathbf{v}$ with the "reflection vector", or a dot project $\mathbf{h}\cdot\mathbf{n}$ with the "halfway vector".

Give the best formulas for the following two vectors (in terms of $\ell, \mathbf{n}, \mathbf{v}$. Express your answers as *unit vectors.*

(a) The reflection vector $\mathbf{r}$.

(b) The halfway vector $\mathbf{h}$.

5. [20 points] A surface of revolution is formed by revolving the function $y = 1/r^2$ around the y-axis. That is, the surface points satisfy $y = 1/(x^2 + z^2)$.

Suppose $\langle x, y, z, \rangle$ is a point on this surface. Give a formula for a vector perpendicular to the surface at the point $\langle x, y, z\rangle$. (It does not need to be a unit vector.) [Hint: There are several ways to work this problem. Some of them are easier than others.]

6. [20 points] Let S be a surface in $\mathbb{R}^2$. Suppose $\mathbf{0} = \langle 0, 0\rangle$ is on the surface S and the normal vector to the surface at this point is $\langle 1, 0\rangle$. Let A be the affine transformation of $\mathbb{R}^2$ defined by

$$A(\mathbf{x}) \;=\; \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}\mathbf{x} \;+\; \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

Thus A is the composition of a translation, a shearing transformation and the scaling S_2. Note $A(\mathbf{0}) = \langle 3, 5\rangle$.

Let $A(S)$ be the surface as transformed by A. Give a vector perpendicular to $A(S)$ at the point $A(\mathbf{0})$. (It does not need to be a unit vector.)

1. [20 points] (Linear interpolation/extrapolation) Let $\mathbf{x} = \langle 2, 0 \rangle$ and $\mathbf{y} = \langle -4, 1 \rangle$.

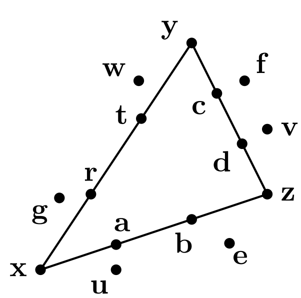
a. What is $\mathbf{u} = \text{LERP}(\mathbf{x}, \mathbf{y}, \frac{1}{4})$? (Give $\mathbf{u}$ explicitly.)

b. What is $\mathbf{v} = \text{LERP}(\mathbf{x}, \mathbf{y}, 2)$. (Give $\mathbf{v}$ explicitly.)

c. Suppose f is a function with $f(\mathbf{u}) = 0$ and $f(\mathbf{v}) = 10$, and we want set f's values for other points by linear interpolation or extrapolation. What will this give for the value of $f(0, \frac{1}{3})$?

d. What point $\mathbf{r}$ on the line containing $\mathbf{x}$ and $\mathbf{y}$ is closest to the origin $\mathbf{0}$? Express your answer in the form $\mathbf{r} = \text{LERP}(\mathbf{x}, \mathbf{y}, \alpha)$ for some value of α.

3. [20 points] Refer to the figure below, for questions about barycentric coordinates $\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z}$. For part b., let $\mathbf{x} = \langle -2, -1 \rangle$ and $\mathbf{y} = \langle 0, 1 \rangle$ and $\mathbf{z} = \langle 1, 0 \rangle$. (Not drawn to scale.)



a. Answer the following questions with the appropriate answer chosen from **a** through **z**:

i. Which point has barycentric coordinates $\alpha{=}\frac{1}{3}$, $\beta{=}0$, $\gamma{=}\frac{2}{3}$?

ii. Which point has barycentric coordinates $\alpha{=}\frac{3}{7}$, $\beta{=}\frac{5}{7}$, $\gamma{=}\frac{-1}{7}$?

b. What are the barycentric coordinates for the point $\mathbf{h} = \mathbf{0} = \langle 0, 0 \rangle$? Draw its approximate location on the figure above.

5. [10 points] A parabolic surface S is defined as the set of points $\langle x, y, z\rangle \in \mathbb{R}^3$ which satisfy $y = -(x^2 + \frac{1}{2}z^2)$. For a general point $\mathbf{u} = \langle x, y, z\rangle$ on S, give the formula for a vector normal to the surface at $\mathbf{u}$. Your vector does not need to be a normal vector, but it should be pointing upward from the surface, not downward.

6. [20 points] Give the formula for the diffuse component of Phong lighting. Also, explain what all the variables represent and any special properties they should satisfy. (For example, do they need to be unit vectors?) Third, draw a picture showing the surface vertex, the light and eye positions, and any relevant vectors. (You should do this for a single color and single light: you do **not** need to describe how to handle multiple lights and multiple colors.)

5. [10 points] A hyperboloid surface $\mathcal{S}$ is defined as

$$\mathcal{S} \;=\; \{\langle x, y, z\rangle : y^2 = 10 + x^2 + 2z^2\}.$$

For $\mathbf{u} = \langle x, y, z\rangle$ a point on $\mathcal{S}$, give an equation for a normal vector $\mathbf{n}$, i.e., normal to $\mathcal{S}$ at the point $\mathbf{u}$. Your vector does not need to a unit vector, but it should be pointing ***towards*** the xz-plane.

1. Let the flattened ellipsoid E have radii 2, 1, 2, and be defined by the equation $x^2 + 4y^2 + z^2 = 4$. Where does the ellipsoid intersect the three axes?

a. Draw a picture by hand, illustrating E as best you can.

b. Suppose $\langle x, y, z\rangle$ is a point on the ellipse. Give a formula for the outward normal at the ellipse at this point.

b. Give a parametric equation for E; that is, a function f of two parameters (for instance, θ and ϕ) so that $f(\theta, \phi)$ gives the points on the ellipsoid as its two parameters vary. What ranges do the parameters need to vary over? *Hint: do this similarly to the parametric description of a sphere in spherical coordinates.*

c. Use the parametric equation for E to give a formula for the outward unit normal at a point on E. Your formula for the normal should be in terms of θ and ϕ.

2. A cone C has tip at $\langle 0, 1, 0\rangle$, and its base is the disk $x^2 + z^2 \leq 1$ in the xz-plane. Suppose $\langle x, y, z\rangle$ is a point on the side of the ellipse (not on the base). Give a formula for the outward unit normal to the cone at this point.

3. Let $\mathbf{x} = \langle -2, 2\rangle$ and $\mathbf{y} = \langle 2, 0\rangle$. Let α control linear interpolation and extrapolation from x to y. What five points are obtained when $\alpha = -2$, $\alpha = 0$, $\alpha = \frac{1}{2}$, $\alpha = 1$, and $\alpha = 2$? What value of α gives the point $\langle 1.6, 0.2\rangle$? Graph by hand all these values, labeling things clearly.

4. Continuing problem 3. If you apply the formula for inverting linear interpolation to the origin $\langle 0, 0\rangle$, what value of α do you get? Use this to compute the point on the line containing $\mathbf{x}$ and $\mathbf{y}$ that is closest to the origin.

5. Let $\mathbf{x} = \langle -1, -2\rangle$, and $\mathbf{y} = \langle 1, 1\rangle$, and $\mathbf{z} = \langle 1, -1\rangle$. What points are obtained by the following sets of barycentric coordinates?

a. $\alpha = 0$, $\beta = 0$, and $\gamma = 1$.

b. $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$, and $\gamma = 0$.

c. $\alpha = \frac{1}{3}$, $\beta = \frac{1}{3}$, and $\gamma = \frac{1}{3}$.

d. $\alpha = \frac{4}{5}$, $\beta = \frac{1}{10}$, and $\gamma = \frac{1}{10}$.

e. $\alpha = \frac{4}{3}$, $\beta = \frac{2}{3}$, and $\gamma = -1$.

6. For the same triangle as in the previous problem, what are the barycentric coordinates of the following points?

a. $\mathbf{u} = \langle 1, 1\rangle$.

b. $\mathbf{u} = \langle \frac{1}{3}, 0\rangle$.

c. $\mathbf{u} = \langle \frac{1}{2}, -\frac{1}{2}\rangle$.

7. For the same triangle again: Draw a graph showing where the points lie that have barycentric coordinates with $\alpha > 0$, $\beta > 0$, and $\gamma < 0$.

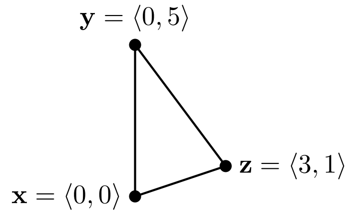
8. Let $\mathbf{x} = \langle 0, 0, 0\rangle$, $\mathbf{y} = \langle 5, 0, 1\rangle$, $\mathbf{z} = \langle 4, 1, 1\rangle$, and $\mathbf{w} = \langle -1, 2, 0\rangle$ be the four vertices of a quadrangle in counterclock-wise order. For each pair of values α and β, what point is obtained by bilinear interpolation in this quadrangle? (Or, if no such point exists, explain why not.)

a. $\alpha = 0$ and $\beta = 1$.

b. $\alpha = 1$ and $\beta = 1$.

c. $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$.

d. $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{3}$.



5. [20 points] Consider the triangle lying in $\mathbb{R}^2$ with vertices $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \langle 0, 5 \rangle$, and $\mathbf{z} = \langle 3, 1 \rangle$.

(a) What point in $\mathbb{R}^2$ has barycentric coordinates $\alpha = \frac{1}{2}$, $\beta = \frac{1}{6}$, and $\gamma = \frac{1}{3}$ relative to this triangle?

(b) What are the barycentric coordinates for the points $\langle 3, 1 \rangle$? For the point $\langle 1, 3 \rangle$?

6. [20 points] A patch $\mathbf{f}(\alpha, \beta)$ in $\mathbb{R}^3$ is defined using bilinear interpolation on the four points $\mathbf{x} = \langle 0, 0, 0 \rangle$, $\mathbf{y} = \langle 6, 0, 3 \rangle$, $\mathbf{z} = \langle 6, 6, 0 \rangle$, and $\mathbf{w} = \langle 0, 6, 0 \rangle$. The points in counterclockwise order around the patch are $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$.

(a) Give the parametric formula $\mathbf{q}(u, v)$ for the patch.

(b) What is the point on this patch with bilinear coordinates $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$?

(c) Give a (non-zero) vector which is normal to the patch at this point. (Your normal vector does not need to be a unit vector.)

4. [20 points]

(a) What does the term "shading" mean in computer graphics?

(b) Briefly describe why shading is important for rendering 3D graphics images.

(c) Briefly describe Gouraud shading and Phong shading and how they are different.

(d) Compare these two kinds of shading. What are the relative advantages and disadvantages of Phong and Gouraud shading?

5. [20 points] Consider the triangle lying in $\mathbb{R}^2$ with vertices $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \langle 1, 3 \rangle$, and $\mathbf{z} = \langle 5, 0 \rangle$.

(a) What point in $\mathbb{R}^2$ has barycentric coordinates $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, and $\gamma = \frac{1}{6}$ relative to this triangle?

(b) What are the barycentric coordinates of the point $\langle 3, 1 \rangle$?

3. (20 points) Barycentric coordinates. Let a triangle in the xy-plane be formed by the three points $\vec{x} = (0, 0)$, $\vec{y} = (3, 0)$ and $\vec{z} = (1, 3)$.

a. Explain (one or two sentences) the definitions of barycentric coordinates for points $\vec{u}$ with respect to this triangle.

b. What are the barycentric coordinates for the point $(1, 3)$?

c. What are the barycentric coordinates for the point $(2, 0)$?

4. (30 points) Give an description of how **specular** reflection is modeled in the Phong reflection model. Include a description of the various vectors and angles and their properties. Include the treatment of multiple light sources. **Include formulas** and explain the terms in the formulas. Explain common shortcuts used in the calculations. You should not include OpenGL specific enhancements such as attenuation and spotlights.

It is best if your description is both complete and succinct. You do not need to explain things at great length, but your answer should make it clear you understand how specular reflection is modeled.

1. [20 points] (Linear Interpolation/Extrapolation.) Let $\mathbf{x} = \langle 1, 2, 0 \rangle$ and $\mathbf{y} = \langle 0, 6, 2 \rangle$. Let L be the line containing $\mathbf{x}$ and $\mathbf{y}$.

a. What is $\textsc{Lerp}(\mathbf{x}, \mathbf{y}, \frac{1}{3})$?

b. What is $\textsc{Lerp}(\mathbf{x}, \mathbf{y}, 1)$?

c. Give the value of α such that $\mathbf{u} = \textsc{Lerp}(\mathbf{x}, \mathbf{y}, \alpha)$ is the point on L which is closest to the origin $\langle 0, 0, 0 \rangle$.

1. [20 points] (Linear interpolation) Let $\mathbf{x} = \langle -2, 0, 0\rangle$ and $\mathbf{y} = \langle 2, 1, 2\rangle$.

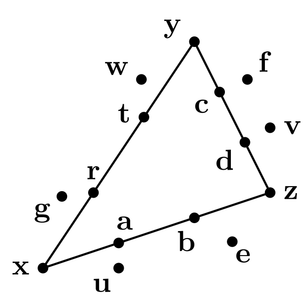
a. What is $\mathbf{u} = \text{LERP}(\mathbf{x}, \mathbf{y}, \frac{1}{4})$? (Give $\mathbf{u}$ explicitly.)

b. What is $\mathbf{v} = \text{LERP}(\mathbf{x}, \mathbf{y}, 1)$. (Give $\mathbf{v}$ explicitly.)

c. Let $\mathbf{z} = \langle 1, \frac{3}{4}, \frac{3}{2}\rangle$. What value α satisfies $\mathbf{z} = \text{LERP}(\mathbf{x}, \mathbf{y}, \alpha)$?

d. Let $\mathbf{w} = \langle -2, 1, 0\rangle$. What point $\mathbf{r}$ on the line containing $\mathbf{x}$ and $\mathbf{y}$ is closest to $\mathbf{w}$?

3. [20 points] Refer to the figure below, for questions about barycentric coordinates $\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z}$. For part b., let $\mathbf{x} = \langle -4, -2\rangle$ and $\mathbf{y} = \langle -2, 1\rangle$ and $\mathbf{z} = \langle 0, 0\rangle$.



a. Answer the following questions with the appropriate answer chosen from **a** through **z**:

i. Which point has barycentric coordinates α=0, β=$\frac{2}{3}$, γ=$\frac{1}{3}$?

ii. Which point has barycentric coordinates α=$\frac{5}{7}$, β=$\frac{-1}{7}$, γ=$\frac{3}{7}$?

b. What are the barycentric coordinates for the point $\mathbf{h} = \langle -1, 0\rangle$? Draw its approximate location on the figure above.

4. [15 points]

a. Give the definition of an "Affine Transformation" mapping $\mathbb{R}^n \to \mathbb{R}^n$.

b. Give the definition of an "Affine Combination" of k points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$.

5. [20 points] Give the formula for the specular component of Phong lighting. Also, explain what all the variables represent and any special properties they should satisfy. (For example, do they need to be unit vectors?) Third, draw a picture showing the surface, the light and the eye position, and the relevant vectors. You do not need to discuss the "halfway vector", just the main Phong formula for specular lighting.

6. [20 points] A smooth surface S in $\mathbb{R}^3$ is transformed by the linear transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ represented by the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 4 & 0 \end{pmatrix}$$

to form the surface $f(S)$. Give a 3×3 matrix B such that whenever $\mathbf{n}$ is normal to S at a point $\mathbf{x}$ on S, then the vector $\mathbf{m} = B\mathbf{n}$ is normal to the point $f(\mathbf{x})$ on the surface $f(S)$. [Hint: It is not difficult to invert A.]

1. [20 points] (Linear interpolation/extrapolation) Let $\mathbf{x} = \langle 2, 0 \rangle$ and $\mathbf{y} = \langle -4, 1 \rangle$.

a. What is $\mathbf{u} = \text{LERP}(\mathbf{x}, \mathbf{y}, \frac{1}{4})$? (Give $\mathbf{u}$ explicitly.)

b. What is $\mathbf{v} = \text{LERP}(\mathbf{x}, \mathbf{y}, 2)$. (Give $\mathbf{v}$ explicitly.)

c. Suppose f is a function with $f(\mathbf{u}) = 0$ and $f(\mathbf{v}) = 10$, and we want set f's values for other points by linear interpolation or extrapolation. What will this give for the value of $f(0, \frac{1}{3})$? (That is, f(<0,1/3>)?)