Math 155A - Computer Graphics - Fall 2022
Midterm Exam — November 15

Name: Answer Key

PID: ....

You must write your initials ON EVERY PAGE. — DO THIS FIRST! This is in case exam pages get separated during the scanning process. At the same time, please check that your copy of the midterm has all 7 problems.

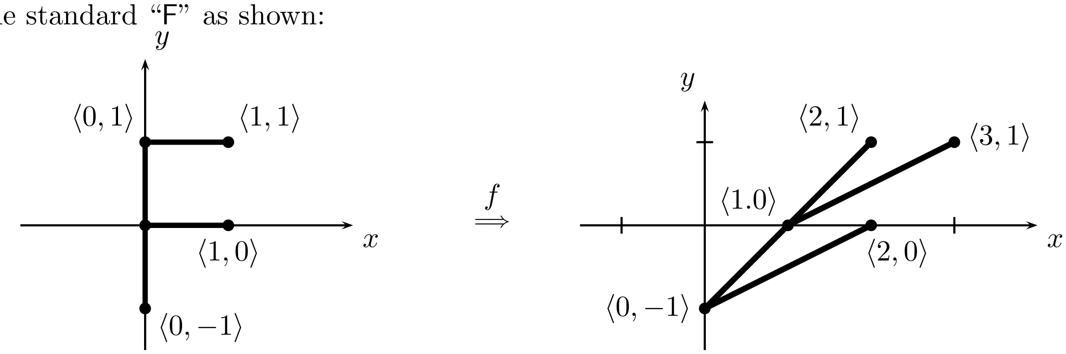
You have 80 minutes. There are 7 problems. You may not use calculators, notes, textbooks, computers, phones, or other resources during this exam.

All answers should be written on the same page as the problem that it answers. Please do not hand in extra pages. Please show enough of your work so that we can see how you obtained your answers.

Use a dark pencil, or a dark blue or black pen so that the test will scan legibly.

Good luck!

1. (Transformation in $\mathbb{R}^2$.) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformation that transforms the standard "F" as shown:



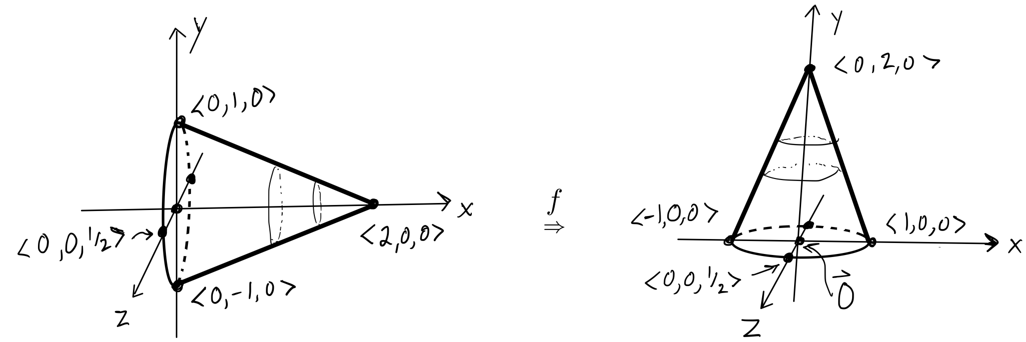
(a) Give a 3×3 matrix that expresses f by acting on homogeneous coordinates.

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Give a 3×3 matrix that expresses the inverse transformation f^{-1} by acting on homogeneous coordinates.

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & -2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

2. (Rotations in $\mathbb{R}^3$.) A rigid and orientation preserving mapping f sends the flattened cone on the left to the flattened cone on the right. The base of the cone is an ellipse with major radius 1 and minor radius $\frac{1}{2}$. The height of the code is 2. Note that the cone is symmetric under a 180° rotation around its central axis: because of this, there two possible choices for the rigid and orientation preserving map f — which we call f_1 and f_2.



(a) Give the 3×3 matrices that represent f_1 and f_2. (There are two matrices since the f shown in the image can be defined in two different ways.)

f_1: $\vec{i} \mapsto \vec{j}$, $\vec{j} \mapsto -\vec{i}$, $\vec{k} \mapsto \vec{k}$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

f_2: $\vec{i} \mapsto \vec{j}$, $\vec{j} \mapsto \vec{i}$, $\vec{k} \mapsto -k$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(b) The two mappings f_1 and f_2 are equal to rotations around the origin. Express them as $R_{\varphi,\mathbf{u}}$ and $R_{\psi,\mathbf{v}}$ by giving $\varphi, \mathbf{u}$ and $\psi, \mathbf{v}$ explictly. (There are multiple possible correct answers.)

f_1: $R_{\pi/2,\, \vec{k}}$

f_2: $R_{\pi,\, \vec{i}+\vec{j}}$

or: $R_{\pi,\, \frac{\vec{i}+\vec{j}}{\sqrt{2}}}$

3. (Rational map.) Suppose $A : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$A(\langle x, y, z\rangle) = \left\langle \frac{x+3y}{x-3y}, \frac{1}{x-3y}, \frac{z}{x-3y} \right\rangle + \langle 1, 0, 0\rangle.$$

Give a 4×4 matrix that acts on homogeneous coordinates to represent A.

$$\langle x, y, z, 1\rangle \mapsto \langle 2x, 1, z, x-3y \rangle$$

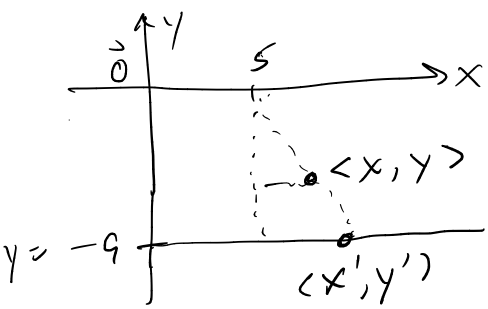
$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix}$$

4. (Shadows.) A light source is placed at $\langle 5, 0, 0\rangle$ and it casts shadows onto the plane P defined by $y = -9$. This plane is horizontal, parallel to the xz-plane.

When $\langle x, y, z\rangle$ is a point in $\mathbb{R}^3$ with $-9 \leqslant y < 0$, define $A(\langle x, y, z\rangle)$ to be the position of the shadow of the point on the plane P. For example, $A(\langle 0, -3, 2\rangle) = \langle -10, -9, 6\rangle$, and $A(\langle 5, -5, 0\rangle) = \langle 5, -10, 0\rangle$.

-9

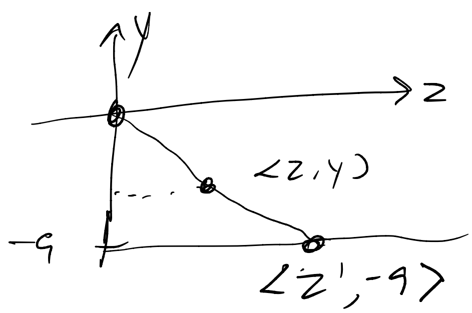
a. Working in ordinary coordinates (not homogeneous coordinates) give the formula expressing the mapping $A(\langle x, y, z\rangle) = \langle x', y', z'\rangle$. That is, give formulas for x', y', z' in terms of x, y, z.



$$y' = -9$$

$$\frac{x-5}{-y} = \frac{x'-5}{9} \Rightarrow x' = 5 - 9\,\frac{x-5}{y}$$

$$x' = \frac{5y - 9x + 45}{y}$$



$$\frac{z}{-y} = \frac{z'}{9} \qquad z' = -\frac{9z}{y}$$

$$\langle x, y, z\rangle \longmapsto \left\langle \frac{-9x + 5y + 45}{y}, -9, \frac{-9z}{y} \right\rangle$$

b. Give a 4×4 matrix that represents the transformation A over homogeneous coordinates.

$$\langle x, y, z, 1\rangle \mapsto \langle -9x + 5y + 45, -9y, -9z, y\rangle$$

$$\begin{pmatrix} -9 & 5 & 0 & 45 \\ 0 & -9 & 0 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(Negative of this matrix, or any non-zero multiple is OK too.)

5. (Normals of a surface.) Let the surface $\mathcal{S}$ be equal to the graph of the function

$$y = (x^3 - x)z.$$

Give a formula for a vector $\mathbf{n}$ which is normal to a point on the surface $\mathcal{S}$. The vector $\mathbf{n}$ should be oriented so that it is pointed generally upwards. Your answer does not need to make $\mathbf{n}$ be a unit vector; it can be given in terms of x, y, z or in terms of just x and z.

$\langle -3x^2z + z, 1, -x^3 + x \rangle$

(or any positive multiple)

6. (Linear interpolation.)
Let $\mathbf{x} = \langle -1, 1 \rangle$. Let $\mathbf{y} = \langle 3, 3 \rangle$. Let L be the line containing $\mathbf{x}$ and $\mathbf{y}$.

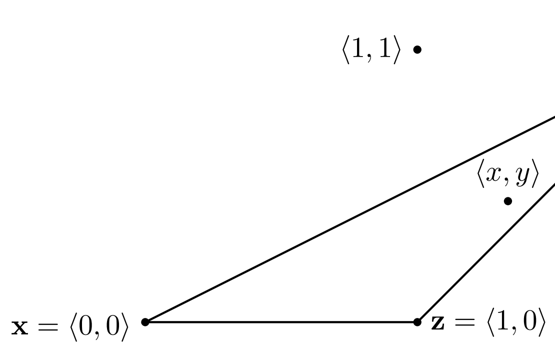
(a) What is $\text{LERP}(\mathbf{x}, \mathbf{y}, -1)$ equal to?

$\langle -5, -1 \rangle$

(b) Let $\mathbf{u}$ be the point on the line L which is closest to the origin. Find the value α so that $\mathbf{u} = \text{LERP}(\mathbf{x}, \mathbf{y}, \alpha)$.

$\alpha = \frac{1}{10}$

7. (Barycentric coordinates.) Consider the triangle in $\mathbb{R}^2$ with vertices $\mathbf{x} = \langle 0,0\rangle$, $\mathbf{y} = \cancel{\langle 2,2\rangle}$ $\langle 2,1\rangle$ and $\mathbf{z} = \langle 1,0\rangle$, as shown in the figure.



(a) What are the barycentric coordinates of $\langle 1,1\rangle$?

$\alpha = 1$

$\beta = 1$

$\gamma = -1$

(b) Give a formula for the barycentric coordinates of an arbitrary point $\langle x,y\rangle$ in $\mathbb{R}^2$. Your answer will give α, β and γ as functions of x and y.

The mapping $\langle x,y\rangle \mapsto \langle\alpha,\beta\rangle$ will be a linear map on $\mathbb{R}^2$. Give a 2×2 matrix that represents this linear map.

$\langle \beta, \gamma\rangle$

$$D = \frac{1}{2}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = \frac{1}{2}\cdot 1$$

$$C = \frac{1}{2}\begin{vmatrix} x & 2 \\ y & 1 \end{vmatrix} = \frac{1}{2}(x-2y)$$

$$B = \frac{1}{2}\begin{vmatrix} 1 & x \\ 0 & y \end{vmatrix} = \frac{1}{2}y$$

$$\beta = \frac{B}{D} = y \qquad \gamma = \frac{C}{D} = x-2y \qquad \alpha = 1-\beta-\gamma = 1-x-y$$

$$M = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$ by the equations for β and γ.