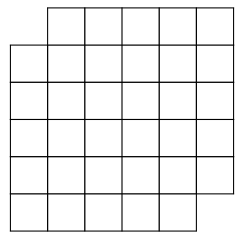
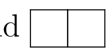
Math 475, Fall 2015
Homework 1
Due: Friday, Sep. 11

(1) Take a 6×6 grid of squares of equal size and remove two diagonally opposite corner squares:



Prove that you cannot tile this with the shapes  and  without any overlaps.

(2) Bóna 1.11: We chose $n + 2$ numbers from the set $1, 2, \ldots, 3n$. Prove that there are always two among the chosen numbers whose difference is more than n but less than $2n$.

(3) Bóna 1.24: Find all 4-tuples (a, b, c, d) of distinct positive integers so that $a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 1.$$

(4) Bóna 2.1: Let $p(k)$ be a polynomial of degree d. Prove that $q(n) = \sum_{k=1}^{n} p(k)$ is a polynomial of degree $d + 1$. Prove that this polynomial q satisfies $q(0) = 0$.
[**Hint:** Define $f^{(0)}(x) = 1$ and for each $d > 0$, define a degree d polynomial

$$f^{(d)}(x) = x(x-1)(x-2)\cdots(x-d+1).$$

(a) Show that the conclusion holds for each $f^{(d)}(x)$ with $d > 0$ by proving that

$$\sum_{k=1}^{n} f^{(d)}(k) = \frac{f^{(d+1)}(n+1)}{d+1}.$$

Verify the conclusion directly for $f^{(0)}(x)$.

(b) Show that if the conclusion holds for polynomials $p_1(x), \ldots, p_r(x)$, then it also holds for any linear combination $\alpha_1 p_1(x) + \cdots + \alpha_r p_r(x)$ (here α_i are scalars).

(c) Show, by induction on d, that any polynomial of degree d is a linear combination of $f^{(0)}(x), f^{(1)}(x), \ldots, f^{(d)}(x)$. For the induction step, note that if c is the leading coefficient of $p(x)$, then $p(x) - cf^{(d)}(x)$ is a polynomial of degree $\leq d - 1$.]

(5) Bóna 2.22: Prove that for all positive integers n,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

(6) Bóna 2.24: Find a closed formula (no summation signs) for the expression

$$\sum_{i=1}^{n} i(i+1).$$

Prove that your formula is correct.