Math 475, Fall 2015
Homework 10
Due: Friday, Dec. 4

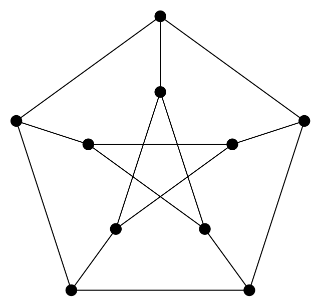
Notation: $[n] = \{1, \dots, n\}$.

Recall from class that the **girth** of a graph is the shortest length of a cycle it contains, and that we proved that if G is a simple planar graph with n vertices, m edges, and finite girth g, then

$$(*) \qquad m \leq \frac{g}{g-2}(n-2).$$

(1) In this exercise, you'll find some examples where (*) is an equality.
  (a) Draw an example of a simple planar graph of girth 3 with 7 vertices and 15 edges.
  (b) For $n \geq 4$, show that the complete bipartite graph $K_{2,n-2}$ is planar, has girth 4, and (*) is an equality.
  (c) Draw an example of a simple planar graph of girth 5 with 8 vertices and 10 edges.

(2) The following graph G is the **Petersen graph**:



  (a) Find 5 edges to contract so that the resulting graph is isomorphic to the complete graph K_5. Conclude that G is not planar.
  (b) Compute the girth of G and show that the inequality (*) fails. (This gives a different reason for the fact that G is not planar.)

(3) If $r \leq n$, an $r \times n$ **Latin rectangle** is an $r \times n$ matrix with entries in $[n]$ such that no row and no column has a repeated entry. In this exercise, you will prove the following theorem of Ryser: if $r < n$, then any $r \times n$ Latin rectangle can be extended to an $(r+1) \times n$ Latin rectangle (meaning that you can add an extra row at the bottom with the numbers $1, \dots, n$ so that the result is an $(r+1) \times n$ Latin rectangle).
  (a) Let $G = (X, Y)$ be a simple bipartite graph such that $|X| \leq |Y|$. Assume that there is a positive integer d such that $\deg(v) = d$ for all vertices v of G. Show that there is a perfect matching of X into Y.
  (b) Given an $r \times n$ Latin rectangle, build a bipartite graph G on $2n$ vertices with $X = Y = [n]$ where $i \in X$ is connected to $j \in Y$ if and only if i does not appear in the jth column of the $r \times n$ Latin rectangle. Show that every vertex of G has degree $n - r$ and use (a) to conclude that G has a perfect matching. Use the perfect matching to construct the next row.

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(4) Let G be a simple planar graph with n vertices and m edges.
  (a) Assume that $\deg(v) \geq 5$ for all vertices v (this implies $n \geq 6$). Use the bound $m \leq 3n - 6$ to prove that $n \geq 12$.
  (b) Use (a) to show that if $m < 30$, then G has a vertex of degree ≤ 4.
  (c) Use (b) and modify the proof of the five-color theorem to show that if $m < 30$, then G has a proper 4-coloring. Don't cite the four-color theorem! (So this proves the four-color theorem for planar graphs with < 30 edges.)

Remark. You can't improve the bounds in #4(a,b) since there is a planar graph with 12 vertices such that every vertex has degree 5 (and thus has 30 edges). To see one, look here: `https://upload.wikimedia.org/wikipedia/commons/8/83/Icosahedron_graph.svg`