Math 475, Fall 2015
Homework 3
Due: Friday, Sep. 25

Notation: $[n] = \{1, \dots, n\}$.

(1) Bóna 5.23: Find the number of compositions of 10 into even pieces (i.e., compositions $(a_1, \dots, a_k)$ of 10 so that each a_i is even).

(2) Bóna 5.27: Fix an integer $n \geq 3$. Find a simple formula for $S(n, n-2)$, i.e., the number of partitions of $[n]$ into $n-2$ parts.

(3) Bóna 5.32: Fix an integer $n \geq 1$. Let $F(n)$ be the number of all partitions of $[n]$ such that every block has size ≥ 2. Prove that $B(n) = F(n) + F(n+1)$, where $B(n)$ is the nth Bell number, i.e., the number of all partitions of $[n]$.

(4) Bóna 4.38: Fix a positive integer n. Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(5) Bóna 4.43: Fix a positive integer n. Show that

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = -\frac{1}{n+1}.$$

[**Hint:** Take an antiderivative of something related to the binomial theorem.]

(6) Fix an integer $n \geq 2$. Call a composition $(a_1, \dots, a_k)$ of n **doubly even** if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, or ...).

Show that the number of doubly even compositions of n is 2^{n-2}.

For example, if $n = 4$, then here are the 4 doubly even compositions of 4:

$$(2,2), \qquad (3,1), \qquad (1,3), \qquad (1,1,1,1).$$

[**Hint:** Find a bijection between the set of doubly even compositions of n and the set of compositions of n which are not doubly even. This shows that exactly half of all compositions of n are doubly even.]