Math 475, Fall 2015
Homework 4
Due: Friday, Oct. 2

Notation: $[n] = \{1, \dots, n\}$.
Convention: $p(0) = 1$, i.e., there is exactly one partition of the integer 0

(1) Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an integer partition and let $\mu = (\mu_1, \dots, \mu_m)$ be the conjugate partition of λ. Prove that

$$\sum_{i=1}^{n} (i-1)\lambda_i = \sum_{j=1}^{m} \frac{\mu_j(\mu_j - 1)}{2}.$$

(2) A consequence of the binomial theorem is

$$(n-1)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} n^{n-j}.$$

Find an alternative explanation for this identity by counting the number of functions $f\colon [n] \to [n]$ that satisfy $f(i) \neq i$ for all $i = 1, \dots, n$ in two different ways (directly, and using inclusion-exclusion).

(3) Bóna 7.27: How many positive integers ≤ 1000 are neither perfect squares nor perfect cubes? [Recall that a perfect square is an integer of the form n^2 where n is an integer, and a perfect cube is an integer of the form n^3 where n is an integer.]

(4) Let λ be an integer partition. Write $\lambda \subseteq m \times n$ if $\ell(\lambda) \leq m$ and $\lambda_1 \leq n$, i.e., the Young diagram of λ fits inside of a $m \times n$ rectangle. For $0 < k < n$, define the **q-binomial coefficient** by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}.$$

In other words, $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in the variable q, where the coefficient of q^i is the number of partitions of i whose Young diagram fits into the $k \times (n-k)$ rectangle. By convention, $\begin{bmatrix} n \\ n \end{bmatrix}_q = \begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. As an example, $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$ (the 1 corresponds to the fact that there is a single partition of size 0).

(a) Show that $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$.

(b) If $0 < k < n$, show that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

(continued on next page)

(c) Using (b), show that plugging in $q = 1$, the q-binomial coefficient becomes the ordinary binomial coefficient. In symbols, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$ for all $0 \leq k \leq n$.
[If you cannot solve (b), you can still use it to solve this problem for credit.]

(d) Find a direct explanation for why $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$. In other words, show that the number of Young diagrams that fit inside the $k \times (n-k)$ rectangle is $\binom{n}{k}$.
[**Hint:** Given a Young diagram $Y(\lambda) \subseteq k \times (n-k)$, we can remove it, and the top boundary of the resulting shape is a path from the bottom left corner of the rectangle to the top right corner using the steps "up" and "right". Show these are counted by $\binom{n}{k}$.]