Math 475, Fall 2015
Homework 5
Due: Friday, Oct. 16

(1) Define a sequence by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 3 \\ a_n &= 8a_{n-1} - 16a_{n-2} \qquad \text{for } n \geq 2. \end{aligned}$$

(a) Express the ordinary generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$ as a rational function in x.
(b) Find a closed formula for a_n.

(2) Define a sequence of numbers $a_0, a_1, \ldots$ by

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 2 \\ a_n &= -a_{n-1} + 2\sum_{i=0}^{n-2} a_i a_{n-2-i} \qquad \text{for } n \geq 2. \end{aligned}$$

Find a simple expression for the ordinary generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$. (You do not need to find a formula for the a_n.)

(3) If $\sum_{n\geq 0} a_n x^n = \dfrac{1 + x + 3x^3}{(1-2x)^4}$, find a formula for the a_n.

(4) Let a_n be the number of partitions of n in which all of the parts are odd, and each number appears ≤ 3 times. By convention, $a_0 = 1$.
For example, for $n = 6$, we get the partitions $\{(5,1), (3,3), (3,1,1,1)\}$, so $a_6 = 3$.

Let b_n be the number of partitions of n in which every part is different, and none of them is divisible by 4. By convention, $b_0 = 1$.
For example, for $n = 6$, we get the partitions $\{(6), (5,1), (3,2,1)\}$, so $b_6 = 3$.

Use generating functions to show that $a_n = b_n$ for all n.

[**Hint:** The factorization $1 + x^k + x^{2k} + x^{3k} = (1 + x^k)(1 + x^{2k})$ might be helpful.]

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(5) In class, we saw that the number of balanced strings of n pairs of parentheses is the **Catalan number** $C_n = \frac{1}{n+1}\binom{2n}{n}$ using generating functions. Now you will derive this formula avoiding generating functions.

Consider the set of paths from $(0,0)$ to (n,n) using the steps $(1,0)$ and $(0,1)$. In HW4, #6(d) (in the context of Young diagrams inside of a rectangle), we saw that there are $\binom{2n}{n}$ of them. To be precise, a path is an ordered sequence of vectors $(v_1,\dots,v_{2n})$ where each v_i is either $(1,0)$ or $(0,1)$ and $v_1+\cdots+v_{2n}=(n,n)$.

For example, for $n=2$, we draw below the two paths $((1,0),(0,1),(1,0),(0,1))$ and $((1,0),(0,1),(0,1),(1,0))$ by starting at $(0,0)$ and adding the vectors in order:

(The bottom left corner is $(0,0)$ and the top right corner is $(2,2)$.)

A path is **good** if it never goes strictly above the diagonal line $x=y$. In symbols, this means that the partial sums $v_1+\cdots+v_i$ always have the property that the first coordinate is greater than or equal to the second coordinate for any $1\le i\le 2n$. Any other path is **bad**.

For example, the path on the left in the example above is good while the path on the right is bad.

(a) Given a bad path $(v_1,\dots,v_{2n})$, let r be the smallest index such that $v_1+\cdots+v_r$ is above the line $x=y$, i.e., the second coordinate is strictly bigger than the first coordinate. Create a new path $(w_1,\dots,w_{2n})$ by

$$w_i = \begin{cases} v_i & \text{if } 1\le i\le r \\ (1,1)-v_i & \text{if } r+1\le i\le 2n \end{cases}.$$

In the example of a bad path above, $r=3$ and the new path w is $((1,0),(0,1),(0,1),(0,1))$.

[In words, w is the same path as v for the first r steps, but then we swap all of the remaining steps. Geometrically, we are reflecting the rest of the path across the line $y=x+1$.]

Show that $w_1+\cdots+w_{2n}=(n-1,n+1)$.

(b) In (a) we defined a function from the set of bad paths to the set of paths from $(0,0)$ to $(n-1,n+1)$. Show that this function is a bijection. Conclude (using HW4, #6(d)) that the number of bad paths is $\binom{2n}{n+1}$, and hence the number of good paths is $\frac{1}{n+1}\binom{2n}{n}$.

(c) Find a bijection between the set of good paths from $(0,0)$ to (n,n) and the set of balanced strings of n pairs of parentheses. To prove correctness of your bijection, do **not** use the fact that the number of balanced strings is $\frac{1}{n+1}\binom{2n}{n}$.

[**Hint:** Given a string S of n pairs of parentheses, define $L_i(S)$ to be the number of left parentheses that are in the first i symbols, and define $R_i(S)$ to be the number of right parentheses that are in the first i symbols (so $L_i(S)+R_i(S)=i$ and $L_{2n}(S)=R_{2n}(S)=n$). Prove that S is balanced if and only if $L_i(S)\ge R_i(S)$ for all $1\le i\le 2n$.]