Math 475, Fall 2015
Homework 6
Due: Friday, Oct. 23

Notation: $[n] = \{1, 2, \ldots, n\}$ and $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$.

(1) Let n be a positive integer and let a_n be the number of different ways to pay n dollars using only 1, 2, 5, 10, 20 dollar bills with the requirement that one cannot use more than five 20 dollar bills. Find a simple expression for the ordinary generating function

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

(2) Let a, b be real numbers. Give a combinatorial proof of the identity

$$e^{(a+b)x} = e^{ax}e^{bx}.$$

More specifically, show that both sides represent the same formal power series by showing that

$$\sum_{n\geq 0} \frac{((a+b)x)^n}{n!} = \left(\sum_{n\geq 0} \frac{(ax)^n}{n!}\right)\left(\sum_{n\geq 0} \frac{(bx)^n}{n!}\right).$$

(3) Let n be a positive integer. Given a group of n people, we want to divide them into nonempty committees and choose a chair and co-chair for each committee (they must be different people). Let h_n be the number of ways to do this and set $h_0 = 1$. Give a simple expression for the exponential generating function

$$H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n.$$

(4) Recall that a derangement of a set S is a bijection $f : S \to S$ such that $f(i) \neq i$ for $i \in S$. (We previously defined this for $S = [n]$ but there's essentially nothing new with this definition.) Let d_n be the number of derangements of $[n]$, and let

$$D(x) = \sum_{n\geq 0} \frac{d_n}{n!} x^n.$$

(a) Without using the formula for d_n that we derived in class, show that

$$D(x)e^x = \frac{1}{1-x}.$$

[**Hint:** $\frac{1}{1-x}$ is the *exponential* generating function for the sequence $c_n = n!$, i.e., the number of permutations on $[n]$. Use the combinatorial interpretation for products of exponential generating functions.]

(b) Dividing both sides by e^x we get $D(x) = e^{-x}\frac{1}{1-x}$ (we know that $1/e^x = e^{-x}$ because $e^x e^{-x} = e^{0x} = 1$ from Problem 2!). Use the product formula to recover the formula we derived in class:

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

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(5) Let k be a positive integer. Let $a(k)_n$ be the number of ordered partitions of $[n]$ into k blocks such that every block has at least 2 elements. Give a simple expression for the exponential generating function

$$A_k(x) = \sum_{n \geq 0} \frac{a(k)_n}{n!} x^n.$$

[**Hint:** This is like our derivation of the exponential generating function for Stirling numbers. What happens for $k = 1$? $k = 2$?]