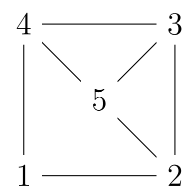
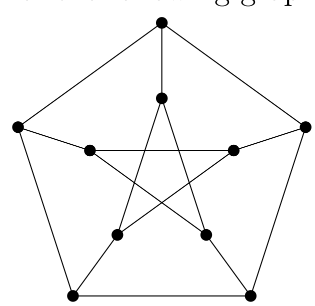
Math 475, Fall 2015
Homework 9
Due: Friday, Nov. 20

(1) How many spanning trees does the following graph have?

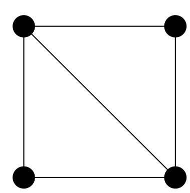


If you use matrix-tree theorem, you may use a computer/calculator to evaluate the determinant.

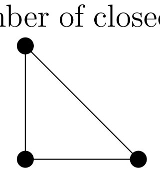
(2) What is the chromatic number of the following graph (with 10 vertices)? Explain.



(3) What is the chromatic polynomial of the following graph?



(4) Let G be a graph and let A_G be its adjacency matrix.
   (a) Show that the number of closed walks of length k in G is trace(A_G^k). (Recall that the trace of a square matrix is the sum of its diagonal entries.)
   (b) Find a simple formula for the number of closed walks of length k in the triangle:



[**Hint:** Recall that the eigenvalues of an $n \times n$ matrix M are defined as follows: let x be a variable, and take the determinant of $xI_n - M$ where xI_n is x multiplied with the $n \times n$ identity matrix. This is a polynomial in x of degree n, and the eigenvalues $\lambda_1, \ldots, \lambda_n$ are the roots of this polynomial (counted with multiplicity). You may use the fact that trace$(M^k) = \lambda_1^k + \cdots + \lambda_n^k$ for all $k \geq 0$.]

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(5) The chromatic polynomial $\chi_G(z)$ of a graph G can be evaluated at any (complex) number z. We so far only have interpretations when z is a non-negative integer (in which case $\chi_G(z)$ is the number of proper z-colorings of G). In this exercise, we will see that $\chi_G(-1)$ counts something related to G.

Define an **acyclic orientation** of G to be a way to put a direction on each edge of G so that it has no directed cycles. Let $\mathrm{AO}(G)$ be the number of acyclic orientations of G. If G has no edges, then we make the convention that $\mathrm{AO}(G) = 1$. (By HW7, #2, $\mathrm{AO}(G) > 0$ for any graph G.) Let n be the number of vertices of G. Prove that

$$\mathrm{AO}(G) = (-1)^n \chi_G(-1)$$

as follows:

(a) If G is a simple graph, let e be an edge of G; show that

$$\mathrm{AO}(G) = \mathrm{AO}(G \setminus e) + \mathrm{AO}(G/e).$$

[**Hint:** Given an acyclic orientation of $G \setminus e$, potentially one of three cases occurs: (1) either way of directing e creates a directed cycle, (2) exactly one way of directing e is an acyclic orientation of G, or (3) both ways of directing e is an acyclic orientation of G.

Show that (1) never happens, in case (2) the acyclic orientation of G has a directed cycle when we contract e, and in case (3), if we contract e, either of the 2 acyclic orientations of G become an acyclic orientation for G/e (in fact, the same one).]

(b) Let $\overline{G}$ be the simple graph obtained from G by, for each pair of vertices x, y, replacing multiple edges between x, y by a single edge $\{x, y\}$ (defined in the deletion-contraction notes). Show that $\mathrm{AO}(G) = \mathrm{AO}(\overline{G})$.

(c) Finish the proof by doing induction on the number of edges of G.