Supplementary notes on planar graphs
Math 475
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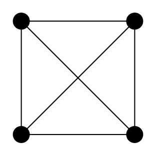
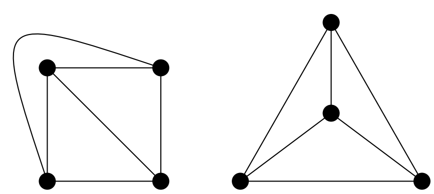
This handout is meant to be a supplement to Section 12.1 of Bóna's book. It takes some work to introduce polyhedra rigorously, and I want to avoid that. They are the topic of Section 12.2 which contains some results about planar graphs, so I'll include those results here. I also want to include some extra results which are useful. Namely, we'll introduce the concept of **girth** which helps one get better bounds on the number of edges of a planar graph. We will also discuss **graph minors**.

1. Definitions

A **planar graph** is, roughly speaking, a graph which can be drawn in the plane (for example, a piece of paper) in such a way that edges do not overlap each other. The basic instructive example is the complete graph K_4. Sometimes it is drawn as follows:



and the two diagonal edges overlap. However, here are two different ways to draw it so that none of the edges overlap:



In the left drawing, one of edges is "curved": this is allowed. In the right drawing, all of the edges are straight lines. It is a theorem that planar graphs can always be drawn so that the edges are all straight lines, but we won't use this and it takes a bit of effort to prove this, so we won't say anything more about it.

It's easy enough to see that if $\overline{G}$ is the simple graph associated to G, then G is planar if and only if $\overline{G}$ is planar.

When you draw a graph in the plane, you have separated the plane into different regions. Another way to say this: if you delete the graph from the plane, the regions are the different connected pieces. These are usually called **faces**. In the right drawing of K_4 above, the faces are the 3 inside triangles and then there is 1 "outside" face, so 4 in total. You can also see there are 4 faces in the left drawing. We'll see shortly that the number of faces only depends on the isomorphism type of the graph.

Once you draw a graph G in the plane, there is something called the **dual graph** G^*: the vertices are the faces of G, and two faces are connected by an edge if they share an edge. This notion is useful for translating the problem of coloring maps into problems of coloring graphs (the countries are faces), but otherwise we won't do much with this definition.

2. Some equations and inequalities

Theorem 2.1 (Euler)**.** *Let G be a connected planar graph with n vertices, m edges, and f faces. Then $n - m + f = 2$.*

This is Theorem 12.2 of Bóna so we don't prove this again here.

Corollary 2.2. *Let G be a planar graph with c connected components, n vertices, m edges, and f faces. Then $n - m + f = c + 1$.*

Proof. We'll do induction on c, the case $c = 1$ being Theorem 2.1. Let G be a graph with c connected components $G_1, \ldots, G_c$. Say G_c has n' vertices, m' edges, and f' faces. Let H be the result of removing G_c. Then H is still planar and has $n - n'$ vertices, $m - m'$ edges, but $f - (f' - 1)$ faces: we subtract $f' - 1$ because G_c and H have the same "outside" face. By induction: $n - n' - (m - m') + f - (f' - 1) = c$. By Theorem 2.1, we also know that $n' - m' + f' = 2$. Adding these two equations gives $n - m + f = c + 1$. □

A simple planar graph with n vertices can't have arbitrarily many edges. We can give a bound in terms of n, but can do much better if we use the girth. For $n \geq 3$, let C_n be the cycle graph with n vertices (so it has n vertices $v_1, \ldots, v_n$ and n edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_n, v_1\}$).

Definition 2.3. A simple graph G has **girth** at least g if it does not contain a subgraph isomorphic to $C_3, C_4, \ldots, C_{g-1}$. (If $g = 3$, this condition is vacuous, so every graph has girth ≥ 3.) We say that G has girth equal to g if it has girth at least g and also contains a subgraph isomorphic to C_g. If G has no cycles, then it has infinite girth. □

If H is a subgraph of G, then the girth of H is at least as big as the girth of G.

Theorem 2.4. *Let G be a simple planar graph with n vertices, m edges, and finite girth g. Then*

$$m \leq \frac{g}{g-2}(n-2)$$

Proof. Let c be the number of connected components of G. Since G has finite girth g, the boundary of every face of G has $\geq g$ edges (the boundary of the outside face is the boundary of G). Each edge is on the boundary of at most 2 faces, so we conclude that $2m \geq gf$. By Corollary 2.2,

$$f = c + 1 - n + m \geq 2 - n + m,$$

so $2m \geq g(2 - n + m)$. Rearranging terms we get $g(n-2) \geq (g-2)m$. Now divide by $g - 2$ (note that $g - 2 \geq 1$ because $g \geq 3$ by the way we defined girth). □

If G has infinite girth, then it's a forest, and we already know that $m \leq n - 1$.

Corollary 2.5. *Let G be a simple planar graph with n vertices and m edges. If $n \geq 3$, then $m \leq 3n - 6$.*

Proof. If G is a forest, then $m \leq n - 1 \leq 3n - 6$ (the second inequality holds because $n \geq 3$). Otherwise, G has finite girth $g \geq 3$ so $m \leq \frac{g}{g-2}(n-2)$ by Theorem 2.4. But $\frac{g}{g-2} \leq 3$, so we can simplify that to $m \leq 3(n-2)$. □

The following result will be useful when we study colorings of planar graphs:

Corollary 2.6. *If G is a simple planar graph, then it has a vertex with degree ≤ 5.*

Proof. Suppose not. Then every vertex of G has degree ≥ 6 (in particular, G has at least 7 vertices). Let m be the number of edges and n be the number of vertices. By the handshake lemma, $2m = \sum_v \deg(v) \geq 6n$ where the sum is over all vertices v. In particular, $m \geq 3n$, which contradicts Corollary 2.5 since $n \geq 7$. $\square$

3. Obstructions to planarity

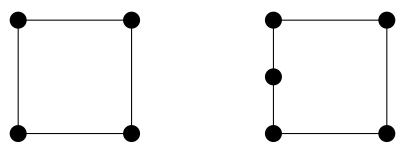
Recall that K_n is the complete graph on n vertices: every pair of vertices has an edge between them. Also recall the complete bipartite graph $K_{n,m}$. It has vertices $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ and the edges $\{x_i, y_j\}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$.

Here are two important examples of non-planar graphs:

Example 3.1. K_5 is not planar: it has girth 3, 5 vertices, and 10 edges. If it were planar, then Theorem 2.4 implies $10 \leq 3(5-2)$ which is false. $\square$

Example 3.2. $K_{3,3}$ is not planar: it has girth 4 (it has no odd cycle because it is bipartite and x_1, y_1, x_2, y_2 is a 4-cycle), 6 vertices, and 9 edges. If it were planar, then Theorem 2.4 implies $9 \leq 2(6-2)$ which is false. Corollary 2.5 only gives a bound of 9 on the number of edges, which isn't good enough to show that $K_{3,3}$ is not planar. $\square$

Given a graph G and an edge $e = \{x, y\}$, the **subdivision** of G along e is a new graph obtained as follows: add a new vertex z, and edges $\{x, z\}$ and $\{y, z\}$ and remove e (pictorially, we've replaced e with two edges. Here's an example where we're subdividing the left edge:



Starting with any graph G, we can subdivide edges all we like (including the new ones), the resulting set of graphs are called subdivisions of G.

It's clear that if G isn't planar, then neither is any subdivision of it (all we really did is add vertices along edges). Furthermore, if G contains a subgraph which isn't planar, then G also can't be planar.

Combining what we know, if G contains a subgraph which is isomorphic to a subdivision of K_5 or $K_{3,3}$, then it isn't planar. The converse is also true, but we won't prove it:

Theorem 3.3 (Kuratowski, 1930)**.** *A simple graph G is planar if and only if it does not have a subgraph which is isomorphic to a subdivision of K_5 or $K_{3,3}$.*

There is a variant which is also convenient for testing planarity. Given a graph G and an edge e, we can delete it to get $G \setminus e$ or contract it to get G/e. We will call both of these **graph minors** of G, and more generally, a graph minor of G is any graph which can be obtained from G by repeatedly deleting edges and vertices, and also contracting edges.

A moment's thought tells us that if G has a graph minor which is non-planar, then G also can't be planar. So we conclude that if G has a graph minor isomorphic to K_5 or $K_{3,3}$, then G is not planar. Again, the converse is also true, but we won't prove it:

Theorem 3.4 (Wagner, 1937)**.** *A simple graph G is planar if and only if it does not have a graph minor which is isomorphic to K_5 or $K_{3,3}$.*