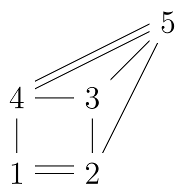
Math 154, Winter 2019
Homework 5
Due: Tuesday, Feb. 19 by noon in basement of AP&M

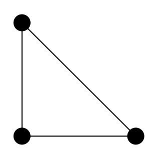
(1) How many spanning trees does the following graph have?



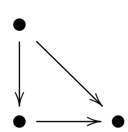
(2) Up to isomorphism, there are 6 different trees with 6 vertices. Draw all of them. Justify why they aren't isomorphic to each other (try to find a unique property that each one has that the others don't).

(3) Let G be a graph and x, y be two vertices which have at least 2 different paths between them. Prove that G has a cycle (this is a step in a proof from class, so you're asked to fill in the missing detail).

(4) Let G be a graph and let A_G be its adjacency matrix.
   (a) Show that the number of closed walks of length k in G is $\mathrm{trace}(A_G^k)$. (Recall that the trace of a square matrix is the sum of its diagonal entries.)
   (b) Use (a) to give a formula for the number of closed walks of length k in the triangle:



(5) Let G be a graph without loops. Define a **cycle-free orientation** of G to be an assignment of directions on each edge of G so that it has no directed cycles. For example, here is a cycle-free orientation of the triangle:



Let $\Omega(G)$ be the number of cycle-free orientations of G. For the triangle, we have $\Omega(G) = 6$ (there are $2^3 = 8$ orientations total and 2 of them are directed cycles). If G has no edges, then we make the convention that $\Omega(G) = 1$.
   (a) Let H be a graph without loops, and let H' be the simple version of H: H' has the same vertices of H, but if there is at least one edge between two vertices x and y in H, we replace all of them with a single edge in H'. Show that
$$\Omega(H) = \Omega(H').$$
   (b) Suppose G is simple. If e is an edge of G, show that
$$\Omega(G) = \Omega(G \setminus e) + \Omega(G/e).$$

Hints

4b: The trace of a square matrix is the sum of its eigenvalues (with multiplicity)

5b: Given a cycle-free orientation of $G \setminus e$, potentially one of three cases occurs: (1) either way of directing e creates a directed cycle for G, (2) exactly one way of directing e is a cycle-free orientation of G, or (3) both ways of directing e is a cycle-free orientation of G.

Show that (1) never happens, in case (2) the cycle-free orientation of G has a directed cycle when we contract e, and in case (3), if we contract e, either of the 2 cycle-free orientations of G become a cycle-free orientation for G/e (in fact, the same one).