Math 154, Winter 2019
Homework 7
Due: Monday, Mar. 11 by 5PM in basement of AP&M

(1) Let G be a simple bipartite graph with bipartition (X, Y). Assume that there is a positive integer d such that $\deg(v) = d$ for all vertices v of G. Show that there is a perfect matching of X into Y.

(2) An $r \times n$ matrix ($r \le n$) is called a **partial Sudoku pattern** if it is filled with $1, \dots, n$ such that each number appears exactly once in every row and at most once in every column. If $r < n$, prove that it is possible to add an extra row to an $r \times n$ partial Sudoku pattern to extend it to one of size $(r+1) \times n$ as follows:

Define a bipartite graph with bipartition (X, Y) where $X = [n]$ and Y is the set of columns of the partial Sudoku pattern and where there is an edge between $i \in X$ and $j \in Y$ if i is not in column j. Show this has a perfect matching and use it to define the next row.

(3) There are 40 houses for sale and 7 buyers. Each one has chosen 5 houses they would be happy to buy, and no 2 buyers have the exact same choice (they might have overlap, but no 2 have chosen the exact same 5 houses). Show that it is possible to match each buyer with a house that they picked out.

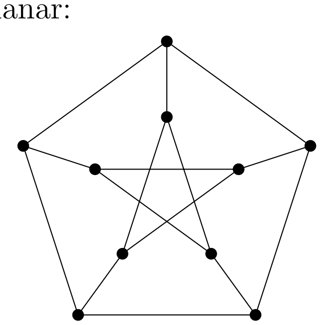
(4) From class: if a simple planar graph has n vertices, m edges, and finite girth $\ge g$, then

$$(*) \qquad m \le \frac{g}{g-2}(n-2).$$

In this exercise, you'll find some examples where $(*)$ is an equality.

(a) Draw an example of a simple planar graph of girth 3 with 7 vertices and 15 edges.
(b) For $n \ge 4$, show that the complete bipartite graph $K_{2,n-2}$ is planar, has girth 4, and $(*)$ is an equality.
(c) Draw an example of a simple planar graph of girth 5 with 8 vertices and 10 edges.

(5) The following graph G is not planar:



Prove this in 2 different ways:

(a) Find 5 edges to contract so that the resulting graph is isomorphic to the complete graph K_5.
(b) Compute the girth of G and show that the inequality $(*)$ from #4 fails.