Math 188, Spring 2021
Homework 3
Due: April 23, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) (a) Let a, b be rational numbers. Show that any non-negative integer n, we have

$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}.$$

(b) When a, b are positive integers, give a *combinatorial interpretation* of this identity, i.e., describe a set whose size can be interpreted as either expression.

(2) How many ways can we arrange the letters of: ALGEBRAICCOMBINATORICS ?

(3) Let $f(n) = \sum_{k=0}^{d} f_k n^k$ be a degree d polynomial with rational coefficients such that $f(a)$ is an integer for all non-negative integers a. (The f_k don't have to be integers for this to be true, for example $f(n) = n(n-1)/2$ has this property.)

(a) Show that there exist **integers** $g_0, \dots, g_d$ such that

$$f(n) = \sum_{k=0}^{d} g_k \binom{d+n-k}{d}.$$

(b) Show that the g_k are the coefficients of the numerator of $\sum_{n\ge 0} f(n)x^n$ when expressed as a rational function.

(c) Express $\sum_{n\ge 0}(n^4 - 3n^3 + 1)x^n$ as a rational function.

(4) Let n be a positive integer. Show that the following are given by the Catalan number C_n:

(a) the number of $2 \times n$ "increasing" matrices; a $2 \times n$ matrix is **increasing** if its entries are
- $1, 2, \dots, 2n$, each appearing exactly once,
- increase in each row going from left to right and,
- for each column, the bottom entry is larger than the top entry.

When $n = 3$, here are the increasing matrices:

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix} \quad \begin{bmatrix}1&2&4\\3&5&6\end{bmatrix} \quad \begin{bmatrix}1&2&5\\3&4&6\end{bmatrix} \quad \begin{bmatrix}1&3&4\\2&5&6\end{bmatrix} \quad \begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}$$

(b) the number of permutations σ of $1, \dots, 2n$ such that

(i) $\sigma = \sigma^{-1}$,

(ii) $\sigma(a) \neq a$ for all a,

(iii) If $a < b < c < d$ and $\sigma(a) = c$, then $\sigma(b) \neq d$.

When $n = 3$, here are the permutations (written in 1-line notation $\sigma(1)\sigma(2)\cdots\sigma(6)$):

$$432165, \quad 632541, \quad 216543, \quad 214365, \quad 654321.$$

(5) Consider the following variation of counting balanced parentheses. We have a new symbol $*$. Let a_n be the number of length n strings consisting of left/right parentheses and $*$ such that the result of deleting all of the $*$'s is a balanced set of parentheses ($a_0 = 1$). Let $A(x) = \sum_{n\ge 0} a_n x^n$. Find a nonzero expression $a(x)t^2 + b(x)t + c(x)$, where $a(x), b(x), c(x)$ are polynomials in x, such that plugging in $t = A(x)$ gives 0.

1. Optional problems (don't turn in)

(6) Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with $(n+2)$ vertices is the Catalan number C_n. By convention, the "2-gon" and triangle both have exactly one triangulation and here are the 5 triangulations of a pentagon: