Math 188, Spring 2021
Homework 6
Due: May 14, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let B_n be the set of binary strings of length n, i.e., of words of length n in the alphabet $\{0, 1\}$. We define a simple graph G_n whose vertex set is B_n where two binary strings are connected by an edge if and only if they differ in exactly one position. Let V be the real vector space with basis $\{v_x \mid x \in B_n\}$. We can think of the adjacency matrix A of G_n as a linear operator on V:

$$A \sum_{x \in B_n} c_x v_x = \sum_{x \in B_n} c_x \sum_{\substack{y \text{ such that} \\ \{y,x\} \text{ is an edge}}} v_y.$$

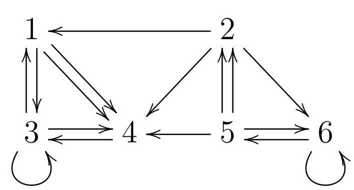
(a) For each $x \in B_n$, define $E_x \in V$ by

$$E_x = \sum_{y \in B_n} (-1)^{x \cdot y} v_y$$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. Show that this is an eigenvector for A with eigenvalue $n - 2|x|$ where $|x|$ is the number of x_i equal to 1.

(b) For $x \in B_n$ and a non-negative integer d, give a formula for the number of closed walks of length d in G_n beginning at x.
[You may assume without proof that $\{E_x \mid x \in B_n\}$ is linearly independent.]

(2) Let A be the adjacency matrix of the following directed graph:



Express $F_{A;1,4}(x)$ as a rational function.

(3) (a) Fix a positive integer k. Construct a directed graph for which walks (between certain vertices) can be interpreted as binary strings with exactly k zeroes. Explain clearly how this interpretation works, including how the length of the walk relates to the length of the binary string.

(b) Construct a directed graph for which walks (between certain vertices) can be interpreted as binary strings in which no symbol ever appears 3 times in a row. Explain clearly how this interpretation works, including how the length of the walk relates to the length of the binary string.

(c) Explain why it is impossible to have a directed graph for which walks of length $2n$ can be interpreted as balanced sets of n pairs of parentheses.

(4) (a) We have n distinguishable tables. We want to paint each one either red, blue, green, or black such that an odd number of them are red and an odd number of them are blue. How many ways can this be done?

(b) Continuing with that situation, we add the colors white and yellow, but the total number of tables which are white or yellow must be even. How many ways are there to choose colors?

(5) Let $a(n,k)$ be the number of set partitions of $[n]$ with k blocks such that no block has size 1. Find a formula for the EGF $A_k(x) = \sum_{n\geq 0} a(n,k)\frac{x^n}{n!}$.

1. Optional problems (don't turn in)

(6) Compute G_3 in Example 4.11.

(7) Given a sequence $(a_n)_{n\geq 0}$ we can define its q-EGF to be

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{[n]_q!}.$$

What is the q-analogue of Proposition 5.2?