Math 188, Spring 2021
Homework 7
Due: May 21, 2021 11:59PM via Gradescope (late penalty waived for this assignment)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

Reminder: The outline for the final project is also due on May 21 via Gradescope. It must be typed, or it will receive 0 credit.

http://www.math.ucsd.edu/~ssam/188/project.html

(1) Let a_n be the number of functions $f\colon [n] \to [n]$ such that $f \circ f = f$. Find a simple formula for the EGF $A(x) = \sum_{n \ge 0} a_n \frac{x^n}{n!}$. Hint at end.

(2) Let $G(x)$ be the unique formal power series such that $[x^n]G(x)^{n+1} = 1$ for all $n \ge 0$. Find a simple formula for $G(x)$. Hint at end.

(3) Given $G(x)$ with $G(0) \ne 0$, define its **logarithmic derivative** to be $\mathcal{L}(G) = \frac{DG(x)}{G(x)}$. By HW2 #3, we have $\mathcal{L}(e^{F(x)}) = DF(x)$ and $\mathcal{L}(G_1(x)G_2(x)) = \mathcal{L}(G_1(x)) + \mathcal{L}(G_2(x))$.

 (a) Let a_n be the number of involutions of size n and let $A(x) = \sum_{n \ge 0} a_n \frac{x^n}{n!}$. From Example 5.11, we have $A(x) = \exp(x + \frac{x^2}{2})$. Apply $\mathcal{L}$ to prove for all $n \ge 0$ that (interpret $a_n = 0$ if $n < 0$)

$$a_{n+1} = a_n + n a_{n-1}.$$

 (b) Let a_n be the number of simple labeled graphs with n vertices where every vertex has degree 2. Use the same method as in (a), but using the formula in Example 5.12, to prove for all $n \ge 0$ that (interpret $a_n = 0$ if $n < 0$):

$$a_{n+1} = n a_n + \binom{n}{2} a_{n-2}.$$

(4) Let $n \ge 1$. Given a labeled tree T with vertices $1, \dots, n$, define $x(T) = x_1^{d_1} \cdots x_n^{d_n}$ where d_i is the degree of vertex i, i.e., the number of edges containing i. Define $\mathbf{C}_n = \sum_T x(T)$ where the sum is over all labeled trees T with vertices $1, \dots, n$. Also define

$$\mathbf{D}_n = x_1 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

 (a) Given a polynomial $p(x_1, \dots, x_n)$, let $p^{(i)}$ be the result of plugging in $x_i = 0$ into the partial derivative $\frac{\partial p}{\partial x_i}$, i.e., the coefficient of x_i if you think of the other variables as constants. If $n \ge 2$, show that

$$\mathbf{C}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{C}_{n-1},$$
$$\mathbf{D}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{D}_{n-1}.$$

 (b) Assuming that $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ show that $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i = 1, \dots, n$.

 (c) Conclude that $\mathbf{C}_n = \mathbf{D}_n$ for all $n \ge 1$.
 [You may use without proof that every tree with at least 2 vertices has a vertex of degree 1.]

1. Optional problems (don't turn in)

(5) $F(x) = \sum_{n \ge 0} f_n x^n$ is a formal power series that satisfies the following identity:

$$F(x) = \exp\left(\frac{x}{2}(F(x) + 1)\right).$$

Find a formula for f_n.

(6) Let n be a positive integer. Given a group of n people, we want to divide them into nonempty committees and choose a leader for each committee, and also choose one of the committees to be in charge of all of the others. Let h_n be the number of ways to do this and set $h_0 = 1$. Give a simple expression for the exponential generating function $H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n$.

(7) Let h_n be the number of bijections $f\colon [n] \to [n]$ with the property that $f \circ f \circ f$ is the identity function. Give a simple expression for the exponential generating function $H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n$.

(8) Let a_n be the number of set partitions of $[n]$ such that every block has at least 2 elements. By convention, $a_0 = 1$. Give a simple expression for the exponential generating function

$$A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n.$$

2. Hints

(1): It may be helpful to think of functions $f\colon [n] \to [n]$ as directed graphs on $[n]$ where an edge $i \to j$ means $f(i) = j$.

(2): Consider the equation $A(x) = xG(A(x))$; from the proof of Lagrange inversion, $G(x)B(x) = x$ where B is the compositional inverse of A.