Linear Recurrence Relations

<u>Setup</u>. Sequence $(a_n)_{n\geq 0} = (a_0, a_1, a_2, \dots)$ satisfies a (homogeneous) linear recurrence relation of order d if $\exists$ constants $c_1, \dots, c_d$ w/ $c_d \neq 0$ s.t.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_d a_{n-d} \quad \forall n \geq d.$$

<u>Example</u> (Fibonacci numbers) $f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$

$f_0 = 0, f_1 = 1 \Rightarrow 0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$

<u>Example</u> <u>$d=1$</u>: $a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = \dots = c_1^n a_0$

Now consider $d=2$: $a_n = c_1 a_{n-1} + c_2 a_{n-2} \quad n \geq 2$ $(c_2 \neq 0)$

<u>Def</u>. The <u>characteristic polynomial</u> is $t^2 - c_1 t - c_2$.

Let r_1, r_2 be its roots $\left(\frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}\right)$

$$\Rightarrow t^2 - c_1 t - c_2 = (t - r_1)(t - r_2) \Rightarrow r_1 \neq 0, r_2 \neq 0.$$

<u>Thm</u>. If $r_1 \neq r_2$, then $\exists$ constants α_1, α_2 s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \forall n \geq 0$$

To solve for α, plug in $n = 0, 1$:

$n=0$: $a_0 = \alpha_1 + \alpha_2$

$n=1$: $a_1 = \alpha_1 r_1 + \alpha_2 r_2$

Example (Fibonacci) $f_0=0, f_1=1, f_n=f_{n-1}+f_{n-2}$ $(n\geq 2)$

char. poly t^2-t-1, $r_1=\frac{1+\sqrt{5}}{2}$, $r_2=\frac{1-\sqrt{5}}{2}$

$\overset{\text{thm}}{\Longrightarrow}$ $f_n=\alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n+\alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ for some α_1, α_2

$\underline{n=0:}$ $0=\alpha_1+\alpha_2 \Rightarrow \alpha_1=-\alpha_2$

$\underline{n=1:}$ $1=\alpha_1 r_1+\alpha_2 r_2 \Rightarrow 1=\alpha_1(r_1-r_2)$

$\Rightarrow \alpha_1=\frac{1}{r_1-r_2}=\frac{1}{\sqrt{5}}$

$$\Rightarrow f_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \forall n\geq 0$$

Example Periodic sequence $x, y, x, y, x, y, \dots$

satisfies $a_n=a_{n-2}$ $\forall n\geq 2$. char. poly is $t^2-1=(t-1)(t+1)$

$\Rightarrow \exists \alpha_1, \alpha_2$ s.t. $a_n=\alpha_1+\alpha_2(-1)^n$ $\forall n\geq 0$

$\underline{n=0:}$ $x=\alpha_1+\alpha_2$

$\underline{n=1:}$ $y=\alpha_1-\alpha_2$

$\Rightarrow \alpha_1=\frac{x+y}{2}$, $\alpha_2=\frac{x-y}{2}$

$$\Rightarrow a_n=\frac{x+y}{2}+(-1)^n\frac{x-y}{2}$$

Rmk. Pick scalars c_1, c_2 s.t. $t^2-c_1t-c_2=(t-r_1)(t-r_2)$ w/ $r_1\neq r_2$ & $c_2\neq 0$. Say a sequence (a_n) is a solution if $a_n=c_1a_{n-1}+c_2a_{n-2}$ $\forall n\geq 2$. Let (a_n') be another solution. Any linear combination $(\gamma a_n+\delta a_n')$ is also a solution.

$\Rightarrow$ Set of solutions is a vector space (subspace of space of all sequences)

Thm says: this subspace is spanned by $(r_1^n), (r_2^n)$

$\Rightarrow$ Solution space is ≤ 2-dim'l

But, $(r_1^n), (r_2^n)$ are linearly independent:

$$(r_1^n) = (1, r_1, r_1^2, \dots) \qquad \Rightarrow \text{ solution space is 2-dim'l}$$
$$(r_2^n) = (1, r_2, r_2^2, \dots)$$

Why are $(r_1^n), (r_2^n)$ solutions?

Have to check that $r_1^n \overset{?}{=} c_1 r_1^{n-1} + c_2 r_1^{n-2}$

$$r_1^n - c_1 r_1^{n-1} - c_2 r_1^{n-2} = r_1^{n-2}(r_1^2 - c_1 r_1 - c_2) = 0$$

Every solution is determined by a_0, a_1

& these can be specified arbitrarily