Set Partitions

Def. X = set, <u>partition</u> of X is an unordered collection of nonempty subsets $S_1, \ldots, S_k$ s.t. every element of X belongs to exactly one S_i.
blocks

An <u>ordered partition</u> is same, except the $S_1, \ldots, S_k$ are ordered.

12-fold: X = set of distinguishable balls. f surjective

k boxes (distinguishable in ordered case
indistinguishable else)

S_i = records ball contained in box #i

Ex. $X = \{1,2,3\}$. There are 5 partitions of X:

$\{\{1,2,3\}\}$ $\quad \{\{1,2\},\{3\}\}$ $\quad \{\{1,3\},\{2\}\}$, $\quad \{\{2,3\},\{1\}\}$ $\quad \{\{1\},\{2\},\{3\}\}$

$\times 2 \qquad \times 2 \qquad \times 2 \qquad \times 6$

ordered partitions $= 1 + 2\cdot 3 + 6 = 13$

Alternative notation

123 $\qquad$ 12|3 $\qquad$ 13|2 $\qquad$ 23|1 $\qquad$ 1|2|3

Ex. 20 pieces of candy (all different)
4 children (clones of each other)

$\left\{\begin{array}{l}\text{Partitions of candy}\\ \text{into 4 blocks}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\text{assignments of candy}\\ \text{to children, so that}\\ \text{no one is empty-handed}\end{array}\right\}$

Def. $S(n,k)$ = # of partitions of set of size n into k blocks
"Stirling number of 2nd kind".

$S(0,0) = 1$. <u>Note</u>: $S(n,k) = 0$ if $k > n$ or if $k = 0$ and $n > 0$

ordered partitions of set of size n into k blocks $= k!\, S(n,k)$

Ex. $n \geq 1$. $S(n,1) = 1$

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$n \geq 2$ $\quad S(n,2) = \dfrac{2^n - 2}{2} = 2^{n-1} - 1$

$S(n, n-1) = \binom{n}{2}$

Thm. $n \geq k \geq 1$. $S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

Pf. Two types of partitions of $X = [n]$:

Type I: Partitions s.t. n is in its own block.

Remove this block: get partition of $[n-1]$ into $k-1$ blocks.

$$\# \text{ type I} = S(n-1, k-1)$$

Type II: Partitions s.t. n shares its block.

Remove n: get partition of $[n-1]$ into k blocks together w/ "special block"

$$\# \text{ type II} = k\,S(n-1, k)$$

$\Rightarrow S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$ $\quad \square$

Small values of $S(n,k)$

$n \backslash k$	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	1	3	1	0	0
4	1	7	6	1	0
5	1	15	25	10	1

Generating function of $S(n,k)$

Fix k, let n vary: $F_k(x) = \sum_{n \geq k} S(n,k)\, x^n$

If $k \geq 1$, $\sum_{n \geq k} S(n,k)\, x^n = x \sum_{n \geq k} S(n-1, k-1) x^{n-1} + kx \sum_{n \geq k} S(n-1,k)\, x^{n-1}$ $\quad m = n-1 \quad m=n-1$

$$= x \sum_{m \geq k-1} S(m,k-1)\, x^m + kx \sum_{m \geq k-1} S(m,k)\, x^m$$

$$= x F_{k-1}(x) + kx F_k(x)$$

$$\Rightarrow \quad F_k(x) = x F_{k-1}(x) + kx F_k(x)$$

$$(1-kx) F_k(x) = x F_{k-1}(x)$$

$$F_k(x) = F_{k-1}(x) \frac{x}{1-kx}.$$

$$F_0(x) = \sum_{n \geq 0} S(n,0)\, x^n = 1 \Rightarrow F_k(x) = \frac{x^k}{(1-kx)(1-(k-1)x) \cdots (1-x)}$$

rational

$\rightsquigarrow$ $\exists$ constants $\alpha_{i,k}$ for $i = 1, \ldots, k$ s.t.

$$S(n,k) = \sum_{i=1}^{k} \alpha_{i,k}\, i^n \quad \text{for } n \geq 1$$

Def. Bell number $B(n) :=$ # partitions of set of size n

By definition, $B(n) = \sum_{k=0}^{n} S(n,k)$

$$\sum_{n \geq 0} B(n) x^n = \sum_{k \geq 0} F_k(x) = \sum_{k \geq 0} \frac{x^k}{(1-x)(1-2x) \cdots (1-kx)}$$

Thm. $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$

Pf. Separate partitions of $[n+1]$ based on how big block containing $n+1$ is. Suppose size is j.

How many such partitions?

Choose $j-1$ numbers from $[n]$ to share w/ $n+1$ $\binom{n}{j-1}$

Choose partition of remaining $n-j+1$ numbers $B(n-j+1)$

$1 \leq j \leq n+1$ are the legal values:

$$B(n+1) = \sum_{j=1}^{n+1} \binom{n}{j-1} B(n-j+1)$$

$i = n+1-j$

$$= \sum_{i=0}^{n} \binom{n}{n-i} B(i) = \sum_{i=0}^{n} \binom{n}{i} B(i)$$

$\square$