Integer Partitions

Def: A partition of an integer n is a sequence of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ such that $\lambda_1 + \cdots + \lambda_k = n$

λ_i = parts of λ. (zero parts are immaterial, only there for convenience)

$|\lambda| = \lambda_1 + \cdots + \lambda_k = n$ (size), $\ell(\lambda) = \#$ of positive λ_i (length)

$p(n) = \#$ partitions of n, $p_k(n) = \#$ partitions of n w/ $\ell = k$

$p_{\leq k}(n) = \#$ partitions of n w/ $\ell \leq k$

12-fold way: integer partitions $\leftrightarrow$ assignments of balls to boxes where boxes & balls both indistinguishable

$p_k(n)$ = surjective assignments of n balls to k boxes

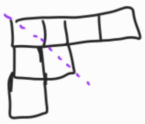
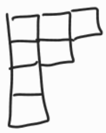
$p_{\leq k}(n)$ = assignments of n balls to k boxes

Note: $p(n) = p_{\leq n}(n)$

Ex. $p(5) = 7$

$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1$

Young diagrams: visualize partitions

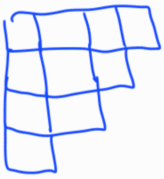
$(4,2,1) \leftrightarrow Y(4,2,1) =$  $\xleftrightarrow{\text{transpose}}$  $= Y(3,2,1,1)$

$(4,2,1)^T = (3,2,1,1)$

$(\lambda^T)_i = |\{ j \mid \lambda_j \geq i \}|$. Note: $(\lambda^T)^T = \lambda$

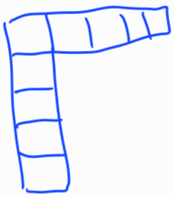
λ is self-conjugate if $\lambda^T = \lambda$.

Ex. Self-conjugate partitions



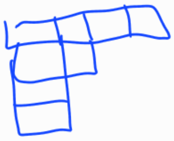
$(4,3,2,1)$

"staircase"



$(5,1,1,1,1)$

"hook"



$(4,2,1,1)$

Thm. # partitions of n w/ $\leq k$ parts $=$ # partitions μ of n s.t. $\mu_1 \leq k$

Pf. Bijection given by taking transpose $\square$

Ex. Consider $\sum_{n\geq 0} p_{\leq k}(n)\, x^n$

Note. $p_{\leq 1}(n) = 1$ for all n, $\sum_{n\geq 0} p_{\leq 1}(n) x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}$

Claim: $\sum_{n\geq 0} p_{\leq k}(n)\, x^n = \prod_{i=1}^{k} \frac{1}{1-x^i} = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$ ✓

Note: $\frac{1}{1-x^i} = 1 + x^i + x^{2i} + x^{3i} + \cdots$

coeff of x^n in $\prod_{i=1}^{k} \frac{1}{1-x^i}$ = # partitions μ of n s.t. $\mu_1 \leq k$

Ex $(4,3,3,1,1,1)$ $k=4$

$\prod_{i=1}^{4}(1-x^i) = (1+x+x^2+\underline{x^3}+\cdots)(\underline{1}+x^2+x^4+x^6+\cdots)(1+x^3+\underline{x^6}+x^9+\cdots)(1+\underline{x^4}+x^8+x^{12}+\cdots)$

Note. $p_{\leq k}(n) = p(n)$ if $k \geq n$

$$\sum_{n\geq 0} p(n) x^n = \lim_{k\to\infty} \sum_{n\geq 0} p_{\leq k}(n)\, x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Let S be subset of positive integers, let $p_S(n) = \#$ partitions μ s.t. all $\mu_i \in S$. Then $\sum_{n\geq 0} p_S(n) x^n = \prod_{i \in S} \frac{1}{1-x^i}$

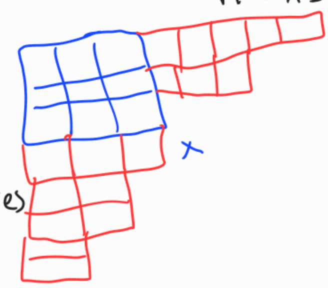
Ex. λ = partition. Durfee square is largest $r \times r$ block in $Y(\lambda)$.

i.e., $r = \max\{i \mid \lambda_i \geq i\}$.

3 pieces to build a partition: (1) Choose r = size of Durfee square

(2) Choose partition w/ $\ell \leq r$ to attach to right of Durfee square

(3) Choose partition w/ all parts $\leq r$ to attach below Durfee square



$$\sum_{n\geq 0} p(n) x^n = \sum_{r \geq 0} x^{r^2} \Big(\sum_{n \geq 0} p_{\leq r}(n) x^n\Big)\Big(\sum_{m\geq 0} p_{\leq r}(m) x^m\Big)$$

$$= \sum_{r \geq 0} \frac{x^{r^2}}{(1-x)^2(1-x^2)^2 \cdots (1-x^r)^2}$$

$p_{odd}(n) = \#$ partitions of n w/ all odd parts

$p_{dist}(n) = \#$ partitions of n w/ all parts different

Thm (Euler) $p_{odd}(n) = p_{dist}(n)$

n=5: $p_{odd}(5) = 3$: $5,\ 3+1+1,\ 1+1+1+1+1$

$p_{dist}(5) = 3$: $5,\ 4+1,\ 3+2$

Pf. $$\sum_{n\geq 0} p_{odd}(n) x^n = \prod_{i \geq 1} \frac{1}{1-x^{2i-1}} = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots}$$

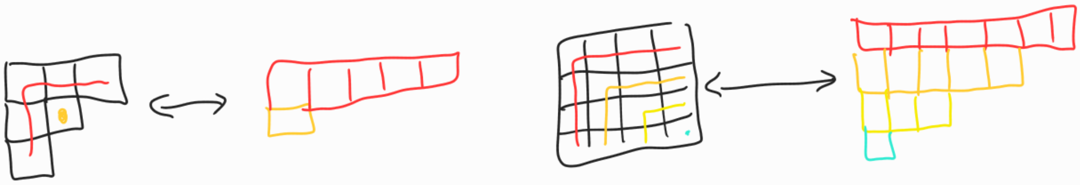
$$\sum_{n \geq 0} p_{dist}(n) x^n = \prod_{i\geq 1}(1+x^i) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

Note: $1 + x^i = \frac{1-x^{2i}}{1-x^i}$

$$\sum_{n\geq 0} p_{dist}(n)x^n = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$
$$= \frac{\cancel{1-x^2}}{1-x}\,\frac{\cancel{1-x^4}}{\cancel{1-x^2}}\,\frac{\cancel{1-x^6}}{1-x^3}\,\frac{\cancel{1-x^8}}{\cancel{1-x^4}}\cdots$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots} = \sum_{n\geq 0} p_{odd}(n)\,x^n \qquad \square$$

Thm. # self-conjugate partitions of n = # partitions of n using distinct odd parts

Pf (sketch)



Given self-conjugate partition $\longrightarrow$ lengths of hooks

λ self-conjugate: length of ith hook $= \lambda_i - (i-1) + \lambda_i^T - (i-1) - 1$
$$= 2\lambda_i - 2i + 1$$

length of ith hook $= 2\lambda_i - 2i + 1 > 2\lambda_{i+1} - 2(i+1) + 1 =$ length of $(i+1)$st hook $\square$

Ex. Generating functions

distinct odd parts: $\prod_{i\geq 1}(1+x^{2i-1}) = (1+x)(1+x^3)(1+x^5)\cdots$

self-conjugate partitions: Choose Durfee square + partition to the right (duplicate it as transpose below)

$$\sum_{r\geq 0} x^{r^2} \sum_{n\geq 0} p_{\leq r}(n)\,x^{2n}$$
$$= \sum_{r\geq 0} \frac{x^{r^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2r})}$$