Transfer Matrix Method

$A = n \times n$ square matrix. Pick $1 \le i, j \le n$. Define

$$F_{A;i,j}(x) = \sum_{k \ge 0} (A^k)_{ij} \, x^k$$

Notation: $B = n \times n$ matrix, $(B; j, i) = (n-1) \times (n-1)$ submatrix obtaining by deleting row j & column i.

Thm. $$F_{A;i,j}(x) = (-1)^{i+j} \frac{\det((\mathrm{id}_n - xA; j, i))}{\det(\mathrm{id}_n - xA)},$$

so $F_{A;i,j}(x)$ is a rational generating function.

Pf. Consider $\sum_{k \ge 0} A^k x^k$ as a $n \times n$ matrix whose entries are formal power series. The inverse of this matrix is $\mathrm{id}_n - xA$. So $F_{A;i,j}(x) = (i,j)$ entry of $(\mathrm{id}_n - xA)^{-1}$

Use Cramer's rule to get desired formula. $\square$

Let $\lambda_1, \ldots \lambda_n$ be eigenvalues of A. Then

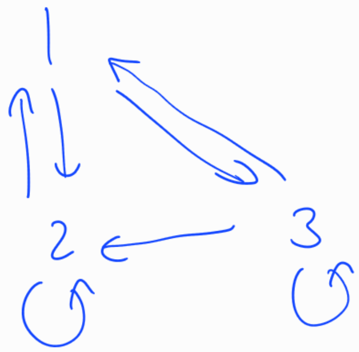
$$\det(t \cdot \mathrm{id}_n - A) = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$$

$t \to 1/x$, multiply by x^n:

$$\det(\mathrm{id}_n - xA) = (1 - \lambda_1 x)(1 - \lambda_2 x) \cdots (1 - \lambda_n x)$$

$\Longrightarrow$ $(A^k)_{ij}$ can be expressed as linear comb. of powers of λ_i^k (for $k \ge n$)

Ex. Consider length n words in $[3]$ s.t. 11 and 23 never appear in consecutive places

$G =$ 

Claim: words of length n $=$ walks of length $n-1$ in G

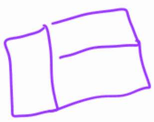
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \det(id_3 - xA) = 1 - 2x - x^2 + x^3$$

Consider words starting w/ 1 and ending at 3.

$$\det((id_3 - xA; 3, 1)) = \det \begin{bmatrix} 1 & -x & -x \\ -x & 1-x & 0 \\ -x & -x & 1-x \end{bmatrix} = x(1-x) = x - x^2$$

$$\Rightarrow F_{A;1,3}(x) = \frac{x - x^2}{1 - 2x - x^2 + x^3} \qquad [x^{n-1}]\, F_{A;1,3}(x) = \#\text{words} \ldots \quad \square$$

Ex. Tile $n \times k$ chessboard by dominoes of size 1×2 or 2×1

such as 

$f_n(k) = \#$ ways to do this.

① If $n=1$, $f_1(k) = \begin{cases} 1 & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd} \end{cases}$

② If $n=2$, consider rightmost column of a tiling.

Two possibilities: either has 1 vertical domino
or 2 horizontal dominoes occupying last 2 columns

$$\Rightarrow f_2(k) = f_2(k-1) + f_2(k-2), \quad f(1) = 1, \ f(2) = 2$$

$\Rightarrow$ Fibonacci numbers.

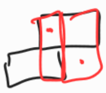
③ If $n=3$, recurrence relation possible, but maybe messy.

<u>General approach for any n</u>: encode tilings as walks in graph G_n

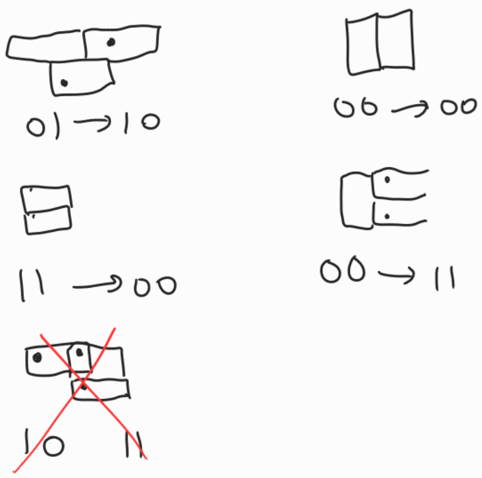
Encode a tiling by marking the squares which are left square of a horizontal domino. Each column becomes word in $\{0,1\}$ of length n: 1 means marked square, 0 means not marked.

Vertices of G_n are words of length n in $\{0,1\}$

Given words w, w', draw edge $w \to w'$ if there is a way to place dominoes so that w appears right before w'.

Eg. $n=2$  $10 \to 01$

$$G_2: \quad \begin{tikzcd} 10 \arrow[r, shift left] & 01 \arrow[l, shift left] \\ 11 \arrow[r, shift left] & 00 \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$



Boundary conditions: rightmost column must be $00\cdots0$
leftmost column has restrictions ...

$\Rightarrow f_n(k) = \#$walks of length $k-1$ starting at valid leftmost column, ending at $00\cdots0$

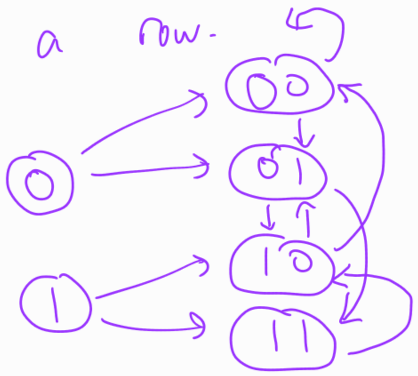
$\Rightarrow \sum_{k\geq 0} f_n(k)x^k =$ finite sum of $F_{A;i,j}(x)$
hence rational

$\Rightarrow f_n(k)$ satisfies linear recurrence relation

Rmk. Words that can be interpreted as walks in a graph

= "regular languages" are recognized by deterministic finite-state automata

Ex. Consider words in $\{0,1\}$ s.t. 1 appears at most twice in a row.



words of length n

= walks of length $n-1$ in this graph starting from 0 or 1.