Products of EGFs

$a_0, a_1, a_2, \ldots$ sequence $\rightsquigarrow A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ $\quad n! = n(n-1)\cdots 2\cdot 1$

<u>exponential generating function</u> (EGF)

Special notation: $e^x = \exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$ (exponential function)

<u>Lemma</u>. $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$, $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$

Then $A(x)B(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$ where $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$

<u>Pf</u>. $\underbrace{[x^n]A(x)B(x)}_{= \frac{c_n}{n!}} = \sum_{i=0}^{n} \frac{a_i}{i!} \frac{b_{n-i}}{(n-i)!} \rightsquigarrow c_n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a_i b_{n-i}$. $\square$

A <u>structure</u> is a function α which takes as input finite sets and outputs finite sets s.t. $|\alpha(S)| = |\alpha(T)|$ if $|S| = |T|$.

<u>Ex</u>. $\alpha(S)$ = set of 2-element subsets of S

= set of set partitions of S

= total orderings of elements of S

$$E_\alpha(x) = \sum_{n\geq 0} |\alpha([n])| \frac{x^n}{n!}$$

Suppose α, β are structures:

(Sum) $(\alpha+\beta)(S) = \alpha(S) \amalg \beta(S)$

(product) $(\alpha\cdot\beta)(S) = \coprod_{T\subseteq S} \alpha(T) \times \beta(S\setminus T)$

<u>Prop</u>. $E_{\alpha+\beta}(x) = E_\alpha(x) + E_\beta(x)$, $\quad E_{\alpha\cdot\beta}(x) = E_\alpha(x)E_\beta(x)$

Pf. $|(\alpha+\beta)([n])| = |\alpha([n])| + |\beta([n])|$ ✓

Products: $$|(\alpha\cdot\beta)([n])| = \sum_{T\subseteq[n]} |\alpha(T)|\cdot|\beta([n]\setminus T)|$$

$$= \sum_{i=0}^{n} \binom{n}{i} |\alpha([i])|\cdot|\beta([n-i])| \quad ✓ \qquad \square$$

Ex. Consider set of n football players.

Split into 2 groups, assign ordering to each group, each member of second group chooses one of 3 colors for uniform.

$c_n = \#$ ways to do this.

- $\alpha(S) =$ set of orderings of elements of S

  $|\alpha(S)| = |S|!$ $\quad E_\alpha(x) = \sum_{n\geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x}$.

- $\beta(S) =$ set of pairs (σ, f), σ is an ordering of S, $f: S\to[3]$

  $|\beta(S)| = |S|!\, 3^{|S|}$ $\quad E_\beta(x) = \sum_{n\geq 0} n!\, 3^n \frac{x^n}{n!} = \frac{1}{1-3x}$.

$c_n = |(\alpha\cdot\beta)([n])|$

$$E_{\alpha\cdot\beta}(x) = E_\alpha(x)\,E_\beta(x) = \frac{1}{(1-x)(1-3x)} = \frac{3/2}{1-3x} - \frac{1/2}{1-x}$$

$$= \frac{3}{2}\sum_{n\geq 0} 3^n x^n - \frac{1}{2}\sum_{n\geq 0} x^n$$

$$\frac{c_n}{n!} = [x^n]E_{\alpha\cdot\beta}(x) = \frac{3}{2}3^n - \frac{1}{2}$$

$$c_n = \frac{n!}{2}(3^{n+1}-1).$$

Ex. We have n distinguishable telephone polls. We paint them red or blue s.t. # blue is even.

C_n = # ways to assign colors.

- $\alpha(S)$ = set of ways to paint polls in S red subject to rules

$|\alpha(S)| = 1$ for all S (even $S = \emptyset$)

- $\beta(S)$ = set of ways to paint polls in S blue subject to rules.

$$|\beta(S)| = \begin{cases} 1 & \text{if } |S| \text{ even} \\ 0 & \text{if } |S| \text{ odd} \end{cases}$$

$$C_n = |\alpha \cdot \beta([n])|$$

$$E_\alpha(x) = \sum_{n \geq 0} \frac{x^n}{n!} = e^x, \quad E_\beta(x) = \sum_{n \geq 0} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2}$$

$$E_{\alpha \cdot \beta}(x) = \frac{1}{2} e^x (e^x + e^{-x}) = \frac{1}{2}(e^{2x} + 1) = \left(\frac{1}{2} \sum_{n \geq 0} \frac{2^n x^n}{n!}\right) + \frac{1}{2}$$

$$e^{A(x)} e^{B(x)} = e^{(A+B)(x)}$$

$$C_n = n! \, [x^n] \, E_{\alpha \cdot \beta}(x) = \begin{cases} \frac{1}{2} n! \frac{2^n}{n!} & \text{if } n > 0 \\ \frac{1}{2} + \frac{1}{2} & \text{if } n = 0 \end{cases}$$

$$= \begin{cases} 2^{n-1} & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}.$$

More than 2 structures. Let $\alpha_1, \dots \alpha_k$ be structures.

$$(\alpha_1 \cdot \alpha_2 \cdots \alpha_k)(S) = \coprod_{\substack{(T_1, \dots, T_k) \\ \text{s.t. } T_1 \cup \dots \cup T_k = S \\ T_i \cap T_j = \emptyset \text{ if } i \neq j}} \alpha_1(T_1) \times \alpha_2(T_2) \times \dots \times \alpha_k(T_k)$$

$$E_{\alpha_1 \cdots \alpha_k}(x) = E_{\alpha_1}(x) \cdots E_{\alpha_k}(x).$$

Ex (cont). Allow green polls (no further rules). How many ways to color red/blue/green? d_n = # ways

$\gamma(S)$ = ways to paint polls in S green subject to rules

$|\gamma(S)|=1$ for all S, $E_\gamma(x)=e^x$

$d_n = |(\alpha\cdot\beta\cdot\gamma)([n])|$

$$E_{\alpha\cdot\beta\cdot\gamma}(x) = \tfrac{1}{2}e^x(e^x+e^{-x})e^x = \tfrac{1}{2}(e^{3x}+e^x) = \tfrac{1}{2}\sum_{n\geq 0}\frac{3^n x^n}{n!} + \tfrac{1}{2}\sum_{n\geq 0}\frac{x^n}{n!}$$

$$\Rightarrow d_n = n!\,[x^n]\,E_{\alpha\beta\gamma}(x) = \tfrac{1}{2}(3^n+1).$$

Ex. Consider structure

$$\alpha(S) = \begin{cases} \{1\} & \text{if } |S|>0 \\ \emptyset & \text{if } S=\emptyset \end{cases}.$$

$$(\alpha\cdot\alpha)(S) = \coprod_{T\subseteq S} \alpha(T)\times\alpha(S\setminus T) = \coprod_{\substack{T\subseteq S \text{ s.t.} \\ T\neq\emptyset \\ S\setminus T\neq\emptyset}} \{1\}$$

if $T=\emptyset$, this term is $\emptyset$

if $S\setminus T=\emptyset$, also get $\emptyset$

$|(\alpha\cdot\alpha)(S)|$ = # ordered partitions of S into 2 blocks $= 2!\,S(|S|,2)$

$|(\alpha^k)([n])|$ = # ordered partitions of S into k blocks $= k!\,S(n,k)$

$$\Rightarrow \sum_{n\geq 0} k!\,S(n,k)\frac{x^n}{n!} = E_{\alpha^k}(x) = E_\alpha(x)^k = (e^x-1)^k$$

$$\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$$