$a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $(c_2 \neq 0)$ $\quad n \geq 2$ $\qquad t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$

$\boxed{\text{Thm: If } r_1 \neq r_2, \text{ then } \exists\, \alpha_1, \alpha_2 \text{ s.t. } a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \ \forall n \geq 0.}$

Pf 1 (Formal power series) $\quad x$ = variable

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n \geq 0} a_n x^n$$

$$A(x) = a_0 + a_1 x + \sum_{n \geq 2} a_n x^n = a_0 + a_1 x + \sum_{n \geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n$$

$$= a_0 + a_1 x + c_1 \sum_{n \geq 2} a_{n-1} x^n + c_2 \sum_{n \geq 2} a_{n-2} x^n$$

$$= a_0 + a_1 x + c_1 x \sum_{n \geq 2} a_{n-1} x^{n-1} + c_2 x^2 \sum_{n \geq 2} a_{n-2} x^{n-2}$$

$m = n-1 \qquad m = n-2$

$$= a_0 + a_1 x + c_1 x \underbrace{\sum_{m \geq 1} a_m x^m}_{A(x) - a_0} + c_2 x^2 \underbrace{\sum_{m \geq 0} a_m x^m}_{A(x)}$$

$$A(x) = a_0 + a_1 x + c_1 x (A(x) - a_0) + c_2 x^2 A(x)$$

$$A(x) - c_1 x A(x) - c_2 x^2 A(x) = a_0 + a_1 x - c_1 a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - c_1 a_0 x}{1 - c_1 x - c_2 x^2} = \frac{a_0 + a_1 x - c_1 a_0 x}{(1 - r_1 x)(1 - r_2 x)}$$

$t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$

$t \to 1/x$

$\frac{1}{x^2} - c_1 \frac{1}{x} - c_2 = (\frac{1}{x} - r_1)(\frac{1}{x} - r_2)$

$1 - c_1 x - c_2 x^2 = (1 - r_1 x)(1 - r_2 x)$

By partial fraction decomposition, $\exists$ constants α_1, α_2 s.t.
(since $r_1 \neq r_2$)

$$A(x) = \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x} \overset{\text{geom. series}}{=} \alpha_1 \sum_{n \geq 0} (r_1 x)^n + \alpha_2 \sum_{n \geq 0} (r_2 x)^n$$

$$\Rightarrow \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} (\alpha_1 r_1^n + \alpha_2 r_2^n) x^n$$

Compare coeffs: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \forall n \geq 0.$ $\square$

Pf 2 (using matrices) $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ becomes

$$\begin{bmatrix} c_1 a_{n-1} + c_2 a_{n-2} \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} \quad \text{for } n \geq 2$$

Set $C = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}$. Note $C^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$ for $n \geq 0$.

Note: If $C = BDB^{-1}$ w/ D diagonal, $C^n = BD^nB^{-1}$

char. poly of $C = \det \begin{pmatrix} t - c_1 & c_2 \\ 1 & t \end{pmatrix} = t^2 - c_1 t - c_2$

$= (t - r_1)(t - r_2)$ $r_1 \neq r_2$

Since $r_1 \neq r_2$, $\exists B$ s.t. $C = B \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} B^{-1}$.

Denote $\begin{bmatrix} x \\ y \end{bmatrix} = B^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$. Then

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = C^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = BD^nB^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = B \begin{bmatrix} r_1^n & 0 \\ 0 & r_2^n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{bmatrix} r_1^n x \\ r_2^n y \end{bmatrix} = \begin{bmatrix} b_{11} r_1^n x + b_{12} r_2^n y \\ b_{21} r_1^n x + b_{22} r_2^n y \end{bmatrix}$$

$$\Rightarrow a_n = b_{21} x r_1^n + b_{22} y r_2^n \quad \forall n \geq 0$$

b_{21}, x, b_{22}, y are constants

Set $\alpha_1 = b_{21} x$, $\alpha_2 = b_{22} y$. $\square$