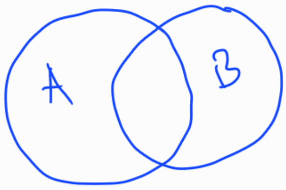
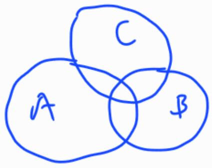
Inclusion - Exclusion

Ex. Two sets A, B

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Three sets A, B, C



$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Thm (Inclusion-Exclusion) $A_1, \dots, A_n$ finite sets

$$|A_1 \cup \dots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \dots < i_j \le n} |A_{i_1} \cap \dots \cap A_{i_j}|$$

Consequence of Möbius inversion for Boolean poset B_n

Lemma. If $S \subseteq T$ are subsets of $[n]$, then

$$\mu(S,T) = (-1)^{|T|-|S|}.$$

Pf. For $S \subseteq T$, suffices to show that

$$\delta_{S,T} = \sum_{U \subseteq [S,T]} (-1)^{|U|-|S|}$$

Set $s = |S|$, $t = |T|$. The number of subsets of size k in the interval $[S,T]$ is $\binom{t-s}{k-s}$. (same as choosing $k-s$ elements in $T \setminus S$)

$$\Rightarrow \sum_{U \subseteq [S,T]} (-1)^{|U|-|S|} = \sum_{k=s}^{t} \binom{t-s}{k-s} (-1)^{k-s} \underset{i=k-s}{=} \sum_{i=0}^{t-s} \binom{t-s}{i} (-1)^i = \delta_{s,t} = \delta_{S,T} \quad \square$$

<u>Pf of Inclusion-Exclusion</u>: Set $A = A_1 \cup \cdots \cup A_n$. For subset $S \subseteq [n]$,

define $f(S) = |\{x \in A \mid x \in A_i \Leftrightarrow i \in S\}|$

$g(S) = |\{x \in A \mid x \in A_i \text{ if } i \in S\}| = \left|\bigcap_{i \in S} A_i\right|$

<u>Note</u>: $g(\emptyset) = |A|$, $f(\emptyset) = 0$, $g(T) = \sum_{T \subseteq S} f(S)$ for all T.

$\xrightarrow{\text{Möbius inversion}}$ $0 = f(\emptyset) = \sum_S g(S)(-1)^{|S|} = \sum_S (-1)^{|S|} \left|\bigcap_{i \in S} A_i\right|$

subtract $g(\emptyset)$ from both sides, multiply by -1:

$$|A| = g(\emptyset) = \sum_{\substack{S \subseteq [n] \\ S \neq \emptyset}} (-1)^{|S|-1} \left|\bigcap_{i \in S} A_i\right| \qquad \square$$

<u>Def.</u> A bijection $f: [n] \to [n]$ is called a <u>derangement</u> if $f(i) \neq i$ for all $1 \leq i \leq n$.

<u>Thm.</u> # derangements of size n is $\sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$.

<u>Pf.</u> For subset $S \subseteq [n]$, define

$f(S) = |\{\text{bijections } f \text{ s.t. } f(i) = i \Leftrightarrow i \in S\}|$

$g(S) = |\{\text{bijections } f \text{ s.t. } f(i) = i \text{ if } i \in S\}| = (n - |S|)!$

derangements $= f(\emptyset)$, $g(T) = \sum_{T \subseteq S} f(S)$

Möbius inversion: $f(\emptyset) = \sum_S (-1)^{|S|} g(S) = \sum_S (-1)^{|S|} (n - |S|)!$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)! = \sum_{j=0}^{n} (-1)^j \frac{n!}{j!}. \qquad \square$$

Rmk. n people w/ hats. Put all hats into a bin, redistribute at random. What is probability no one gets their hat back?

Use calculus! Taylor expansion $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$

Converges everywhere. Plug in $x=-1$: $\frac{1}{e} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$

The n^{th} truncation $\sum_{i=0}^{n} \frac{(-1)^i}{i!}$ is $\frac{\#\text{derangements}}{n!}$ = probability a random bijection is a derangement.

Taylor's Thm tells us $\lim_{n\to\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \frac{1}{e}$

$\Rightarrow$ $\lim_{n\to\infty}$ probability of a derangement $= 1/e \approx .368$

Thm. $n \geq k > 0$, $S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$.

Pf. $k!\, S(n,k) =$ # surjective functions $f: [n] \to [k]$. For $S \subseteq [k]$, define

$f(S) = |\{\text{surjective functions } f: [n] \to S\}|$

$g(S) = |\{\text{functions } f: [n] \to S\}| = |S|^n$

$g(T) = \sum_{S \subseteq T} f(S)$. Möbius inversion $\Longrightarrow$

$$k!\, S(n,k) = f([k]) = \sum_{S \subseteq [k]} g(S)(-1)^{k-|S|} = \sum_{S \subseteq [k]} (-1)^{k-|S|} |S|^n$$

$$= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n. \quad \text{Divide by } k!. \qquad \square$$

Rmk. From before: $\displaystyle\sum_{n\geq k} S(n,k)\,x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$

$\Rightarrow$ $S(n,k)$ is a linear combination of $1^n, 2^n, 3^n, \ldots, k^n$

(for fixed k) coeff of i^n is $\dfrac{(-1)^{k-i}\binom{k}{i}}{k!} = \dfrac{(-1)^{k-i}}{i!\,(k-i)!}$