Burnside's Lemma

$G \curvearrowright X$, $g \in G$

$X^g = \{x \in X \mid g \cdot x = x\}$ fixed points of g

Thm (Burnside)

$$\#\text{ orbits} = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Pf. $S = \{(g,x) \in G \times X \mid g \cdot x = x\}$

For $g \in G$, $(g,x) \in S \iff x \in X^g$

$$\Rightarrow |S| = \sum_{g \in G} |X^g|$$

For $x \in X$, $(g,x) \in S \iff g \in G_x$

$$\Rightarrow |S| = \sum_{x \in X} |G_x|$$

$$\frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{|S|}{|G|} = \sum_{x \in X} \frac{|G_x|}{|G|} = \sum_{x \in X} \frac{1}{|G \cdot x|} = \#\text{orbits} \quad \square$$

Ex. $G = \mathfrak{S}_n$, $X = [n]$ (usual action)

For $\sigma \in \mathfrak{S}_n$, $|X^\sigma| = \#$ cycles of length 1

$$\Rightarrow \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} |X^\sigma| = \text{average } \# \text{ of cycles of length 1 of a permutation}$$

$$\overset{\text{Burnside}}{=} 1$$

Y = finite set, Y^X = set of functions $X \to Y$

G acts on Y^X by $g \cdot f = f \circ \varphi_{g^{-1}}$ for $g \in G$, $f \in Y^X$

$$g \cdot (h \cdot f) = (f \circ \varphi_{h^{-1}}) \circ \varphi_{g^{-1}} = f \circ \varphi_{h^{-1} g^{-1}} = f \circ \varphi_{(gh)^{-1}} = (gh) \cdot f$$

Alternatively, $(g\cdot f)(x) = f(g^{-1}\cdot x)$

$c_X(g) = \#$ cycles of $\varphi_g \in \mathfrak{S}_X$

<u>Thm</u>. $\#$ orbits of G on $Y^X = \dfrac{1}{|G|}\sum_{g\in G} |Y|^{c_X(g)}$.

<u>Pf</u>. By Burnside, suffices to show $|(Y^X)^g| = |Y|^{c_X(g)}$

If $f \in (Y^X)^g$, then $f(g^{-1}\cdot x) = f(x)$ $\forall x \in X$.

$\iff$ f constant on cycles of $\varphi_{g^{-1}}$

$\iff$ f constant on cycles of φ_g

$|Y^X|^g = \#$ functions from cycles of φ_g to Y

$= |Y|^{c_X(g)}$. $\square$

<u>Ex</u>. Counting necklaces. (length n, alphabet size k)

$X = \mathbb{Z}/n$, $G = \mathbb{Z}/n$, $g\cdot x = g + x$.

Y = alphabet of size k

orbits of G on $Y^X \longleftrightarrow$ necklaces of length n w/ alphabet of size k

Consider $n=4$. Elements of G are powers of $(0\ 1\ 2\ 3)$:

	$(0\ 1\ 2\ 3)$,	$(0\ 2)(1\ 3)$,	$(0\ 3\ 2\ 1)$,	$(0)(1)(2)(3)$
#cycles	1	2	1	4

$\Rightarrow \#$ necklaces $= \dfrac{1}{4}(2k + k^2 + k^4)$

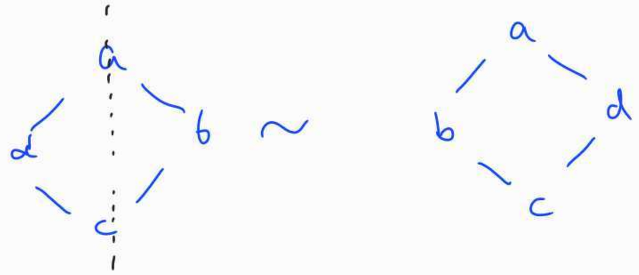
General n? $G =$ powers of $(0\ 1\ 2\ \cdots\ n-1)$

Recall: $(0\ 1\ 2 \cdots n-1)^i$ has order $\dfrac{n}{\gcd(n,i)}$

w/ $\gcd(n,i)$ many cycles

$\Rightarrow$ # necklaces $= \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} k^{\gcd(n,i)}$

Ex. Consider necklaces equivalent if differ by reflection.



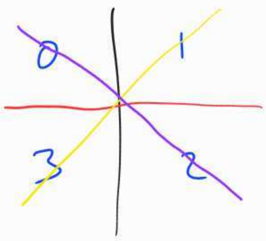
$G = D_n$ dihedral group (size $2n$)

$X =$ vertices of regular n-gon

$Y =$ alphabet of size k

n = 4: vertices $= \{0, 1, 2, 3\}$

get powers of $(0\ 1\ 2\ 3)$ as before $(\mathbb{Z}/n \subset D_n)$



	# cycles
$(0)(2)(1\ 3)$	3
$(1)(3)(0\ 2)$	3
$(0\ 1)(2\ 3)$	2
$(0\ 3)(1\ 2)$	2

necklaces up to reflection $= \dfrac{1}{8}\left(2k + 3k^2 + 2k^3 + k^4\right)$