Choice Problems

$[n] = \{1, \ldots, n\}$. Count subsets of $[n]$ of size k.

Consider expansion of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$

Choice of 1 or x at each step $\longleftrightarrow$ subset of $[n]$ of size k where x is chosen k times

$n=5$ $\quad (1+\underline{x})(\underline{1}+x)(\underline{1}+x)(1+\underline{x})(1+\underline{x}) \longleftrightarrow \{1,4,5\} \subseteq [5]$

Prop. $\#$subsets of $[n]$ of size $k = [x^k](1+x)^n = \binom{n}{k}$

Pascal's identity: $[x^k](1+x)^n = [x^k](1+x)^{n-1}(1+x)$

$$\binom{n}{k} \overset{\text{Pascal}}{=} \binom{n-1}{k} + \binom{n-1}{k-1}$$

Our binomial thm: $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Homogenize: $x \mapsto \frac{x}{y}$: $\quad \boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$

Multiply by y^n

Cor. $\#$subsets of $[n] = 2^n$

Pf. $\#$subsets of $[n] = \sum_{k=0}^{n} \#$subsets of size k of $[n]$

$$= \sum_{k=0}^{n} [x^k](1+x)^n = \text{sum of coefficients of } (1+x)^n$$

Plug in $x=1$: $\quad 2^n$ $\square$

Ex. Substitute $x=2, y=3$ into binomial thm: $5^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 3^{n-k}$.

Take derivative: $\quad n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$

$x \to 1$: $\quad \boxed{n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}}$

$x_1, \dots, x_d$ variables

Given $k_1 + \cdots + k_d = n$ ($k_i \geq 0$ integers), define

<u>multinomial coefficient</u> $\displaystyle\binom{n}{k_1, k_2, \dots, k_d} = \frac{n!}{k_1! k_2! \cdots k_d!}$

<u>Thm</u> (Multinomial Thm) $\displaystyle (x_1 + \cdots + x_d)^n = \sum_{k_1 + \cdots + k_d = n} \binom{n}{k_1, \dots, k_d} x_1^{k_1} \cdots x_d^{k_d}$

sum over all triples $(k_1, \dots, k_d)$ s.t. $k_i \geq 0$ integers & $k_1 + \cdots + k_d = n$

<u>Pf</u>. Induction on d. If $d=1$, both sides are x_1^n ✓

Now assume $d > 1$, multinomial thm holds for $d-1$ variables.

Substitute $x \to x_1 + \cdots + x_{d-1}$, $y \to x_d$ into binomial thm:

$$(x_1 + \cdots + x_d)^n = \sum_{m=0}^{n} \binom{n}{m} (x_1 + \cdots + x_{d-1})^m x_d^{n-m}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \sum_{k_1 + \cdots + k_{d-1} = m} \binom{m}{k_1, \dots, k_{d-1}} x_1^{k_1} \cdots x_{d-1}^{k_{d-1}} x_d^{n-m}$$

set $k_d = n-m$: note $\displaystyle\binom{n}{n-k_d}\binom{n-k_d}{k_1, \dots, k_{d-1}} = \frac{n!}{\cancel{(n-k_d)!}\, k_d!} \frac{\cancel{(n-k_d)!}}{k_1! \cdots k_{d-1}!}$

$\displaystyle = \binom{n}{k_1, \dots, k_d}$

$$= \sum_{k_1 + \cdots + k_d = n} \binom{n}{k_1, \dots, k_d} x_1^{k_1} \cdots x_d^{k_d}. \qquad \square$$

What do multinomial coeff count?

Suppose $1, \dots, d$ represent colors. We have n objects in a row, and we need to assign a color to each.

$(x_1 + \cdots + x_d)^n = (x_1 + \cdots + x_d) \cdots (x_1 + \cdots + x_d)$

<u>Ex</u>. $n=d=3$ $\quad (\underline{x_1} + x_2 + x_3)(x_1 + x_2 + \underline{x_3})(x_1 + x_2 + \underline{x_3}) \to x_1 x_3^2$ $\iff$ coloring first object 1 last two objects 3

Prop. Assume we have d types of objects (colors)

Then $\binom{n}{k_1, \ldots, k_d}$ = # ways to arrange n objects s.t. exactly k_i many objects are of the ith type.

Ex. 10 houses in a row. We have to paint 4 blue, 2 red 3 green, 1 orange. # ways to assign colors is $\binom{10}{4,2,3,1} = \frac{10!}{4!2!3!1!} = 12600$.

A variation: multisets

A multiset of size k of $[n]$ is a choice of k elements, but we can choose elements more than once.

Ex. $\{1,1,1,2,2,3,5\}$ is a multiset of size 7 of $[5]$

A multiset of size k of $[n]$ $\Longleftrightarrow$ a way to multiply out term in $(1+x+x^2+\cdots)^n = \left(\sum_{d\geq 0} x^d\right)^n$

$$1112235 \Longleftrightarrow (1+x+x^2+\underline{x^3}+\cdots)(1+x+\underline{x^2}+\cdots)(1+\underline{x}+x^2+\cdots)(\underline{1}+x+x^2+\cdots)(1+\underline{x}+x^2+\cdots) \searrow x^7$$

Note: $\sum_{d\geq 0} x^d = (1-x)^{-1}$

Prop. # multisets of size k of $[n]$ $= [x^k](1-x)^{-n} = (-1)^k\binom{-n}{k}$
$$= \binom{n+k-1}{k}$$

Suggests: We can find bijection

$$\left\{\begin{array}{c}\text{multisets of size } k \\ \text{of } [n]\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{subsets of size } k \\ \text{of } [n+k-1]\end{array}\right\}$$