Math 188, Fall 2022
Homework 6
Due: November 30, 2022 11:59PM via Gradescope
(late submissions allowed up until December 1, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) The following exercise gives another proof of Cayley's formula, and at the same time provides new information that our proof doesn't give.

Let $n \geq 1$ and let $x_1, \dots, x_n$ be variables. Given a labeled tree T with vertices $1, \dots, n$, define the monomial $x(T) = x_1^{d_1} \cdots x_n^{d_n}$ where d_i is the degree of vertex i, i.e., the number of edges containing i. Define $\mathbf{C}_n = \sum_T x(T)$ where the sum is over all labeled trees T with vertices $1, \dots, n$. Also define

$$\mathbf{D}_n = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

(a) Given a polynomial $p(x_1, \dots, x_n)$, let $p^{(i)}$ be the result of plugging in $x_i = 0$ into the partial derivative $\frac{\partial p}{\partial x_i}$, i.e., the coefficient of x_i if you think of the other variables as constants. If $n \geq 2$, show that

$$\mathbf{C}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{C}_{n-1},$$
$$\mathbf{D}_n^{(n)} = (x_1 + x_2 + \cdots + x_{n-1})\mathbf{D}_{n-1}.$$

(b) Assuming that $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ show that $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i = 1, \dots, n$.

(c) Conclude that $\mathbf{C}_n = \mathbf{D}_n$ for all $n \geq 1$.
[You may use without proof that every tree with at least 2 vertices has a vertex of degree 1.]

(2) How many ways are there to list the letters of the word MATHEMATICS so that no two consecutive letters are the same?

(3) Let $n \geq 2$ be an integer. We have n married couples ($2n$ people in total).

(a) How many ways can we have the $2n$ people stand in a line so that no person is standing next to their spouse?

(b) Same as (a), but replace "line" by "circle".

(4) Let q be a prime power and n a positive integer. Let V be an n-dimensional $\mathbf{F}_q$-vector space and let P be the poset whose elements are linear subspaces of V with the ordering $X \leq Y$ if X is contained in Y. Show that the Möbius function of P is given by

$$\mu(X, Y) = (-1)^d q^{\binom{d}{2}}$$

where $d = \dim Y - \dim X$. Hint at end.

(5) Let Π_n be the poset of set partitions of $[n]$ and let μ be its Möbius function. Write a formula for the number of connected labeled graphs with vertex set $[n]$ using μ (a formula for μ is given in the book, but you don't need to look it up).

Hints

4: For subspaces $X \subseteq Y$, the set of r-dimensional subspaces Z such that $X \subseteq Z \subseteq Y$ are in bijection with $(r - \dim X)$-dimensional subspaces in the quotient space Y/X, and $\dim(Y/X) = \dim Y - \dim X$.

Theorem 3.2.4 from Sagan's book is helpful here.

Optional problems (don't turn in)

(6) $F(x) = \sum_{n\geq 0} f_n x^n$ is a formal power series that satisfies the following identity:

$$F(x) = \exp\left(\frac{x}{2}(F(x)+1)\right).$$

Find a formula for f_n.

(7) Reminder: Lagrange's version of the Taylor remainder theorem says this: if $f(x)$ is an infinitely differentiable function whose Taylor series at 0 converges at $x = r$, then there exists ξ between 0 and r such that

$$f(r) - \sum_{i=0}^{n} \frac{f^{(i)}(0)}{i!} r^i = \frac{f^{(n+1)}(\xi)}{(n+1)!} r^{n+1}.$$

Use the Taylor remainder theorem to show that

$$\left| \frac{1}{e} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right| \leq \frac{1}{(n+1)!}$$

and conclude from this that the number of derangements of n objects is inside the closed interval

$$\left[\frac{n!}{e} - \frac{1}{n+1}, \frac{n!}{e} + \frac{1}{n+1} \right].$$

In particular, show that it is the closest integer to $n!/e$.

(8) Let d_n be the number of derangements of $[n]$, and let

$$D(x) = \sum_{n\geq 0} \frac{d_n}{n!} x^n.$$

(a) Using the structure interpretation for products of EGF, show that

$$D(x)e^x = \frac{1}{1-x}.$$

(b) Show how this implies the formula we previously obtained:

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

(9) For a positive integer n, define

$$f(n) = |\{i \in \mathbf{Z} \mid 1 \leq i \leq n,\ \gcd(n,i) = 1\}|.$$

(a) Show that

$$n = \sum_{d|n} f(d)$$

where the sum is over all positive integers d that divide n.

(b) Use Möbius inversion to show that

$$f(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where the product is over the primes p that divide n.

(10) There are n people sitting at a circular table. How many ways can they rearrange seats so that no one sits next to someone they were sitting next to before?

(11) Let q be a prime power and let N_n be the number of monic irreducible polynomials of degree n with coefficients in $\mathbf{F}_q$.

(a) Using that polynomials over a field satisfy unique factorization, show that

$$(1 - qx)^{-1} = \prod_{d \geq 1} (1 - x^d)^{-N_d}.$$

(b) Take the logarithmic derivative of (a) and compare the coefficient of x^{n-1} to get $q^n = \sum_{d|n} dN_d$.

(c) Use Möbius inversion to get a formula for N_n.